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AN EARTH-CENTERED DATUM
FROM OPTICAL OBSERVATIONS
OF THE EARTH'S HORIZON
FROM AN EARTH SATELLITE

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SUMMARY

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The necessary equations for a theoretical method for obtaining directly an earth-centered world geodetic system from angular measurements involving the earth's visible horizon as seen from an earth satellite are derived. As a check on the method, a computer program was written whereby data could be simulated and results obtained.

The method is found to be mathematically feasible, and the results for a particular case are given to show the accuracy which can be expected under ideal conditions. Further, when large variances in the observations were assumed, the method was found to converge, although slowly. Finally, some of the conditions for the most effective application of the theory are determined.

Although the method is found to be mathematically feasible and to converge for large variances in the measurements, the results are preliminary and a detailed statistical analysis is needed. As for additional applications, the theory can be useful in a variety of theoretical studies, such as a study of the accuracy penalties resulting from assuming the earth to be spherical rather than oblate in navigation schemes and horizon-uncertainty studies.

INTRODUCTION

Author

The technological advances since World War II have instigated a phenomenal growth in interest for an accurate world geodetic system. This increased interest has resulted chiefly from the national objective of space exploration.

In order to complete a deep space probe, it is desirable to know the precise locations of the launch site and of the radar stations which will observe the flight of the space vehicle. The error in the location of a point on the

*The material presented herein is based on a thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Geodetic Science, Ohio State University, Columbus, Ohio, May 1965.

earth's surface is not necessarily a function of the error in the survey alone. Because of the irregular shape of the earth's surface, the variation of the earth's internal density with location, and the uncertainty in the location of the earth's center, the accuracy with which the launch site and radar stations can be located on a particular datum is limited.

Geodesists have developed several sophisticated methods for dealing with each of these factors. Even so, to date, data are insufficient to determine precisely the earth's center and the ellipsoid which will best fit the earth as a whole. Recently, the artificial earth satellite has been employed for geodetic purposes. A major finding has been that the flattening f is approximately $1/298.3$ rather than the $1/297$ previously supposed. Further, the last three digits of the magnitude of the semimajor axis a , previously thought to be 6 378 388 meters, are seriously questioned.

The necessary equations for a theoretical method for obtaining directly an earth-centered world geodetic system will be derived herein. The results for a particular case will be given to show the accuracy which can be expected under ideal conditions. Further, it will be determined if the method will, in fact, converge when large variances in the observations are assumed; finally, some of the conditions for the most effective application of the theory will be determined.

SYMBOLS

A,B,C	coefficients of quadratic equation
Az _g	geodetic azimuth
Az _s	spherical azimuth
a	semimajor axis of earth
b	semiminor axis of earth
c	constant defined in appendix D
D	radial distance from V to T
E	column matrix of residuals
e	eccentricity of earth
F',L,M,S,A,P,Φ	functions (Main symbols are used in subscript position to indicate the independent variable; for example, a is the variable in $\Phi'_{a,n}$.)
f	flattening of earth

H	horizon uncertainty
h	height of surface point above ellipsoid
I	arbitrary point in a plane
N	normal to ellipsoid
NP	North Pole
O	arbitrary observable
p	upper limit of summation
$\ Q\ $	rectangular matrix of coefficients of observation equations
$[q]$	row matrix of coefficients of arbitrary observation equation
R	radial distance from earth's center (to a point)
R_M	mean radius of earth
r	perpendicular distance from earth's polar axis (to a point)
SP	South Pole
s	radar station
T	arbitrary point on earth's visible horizon
$\{U\}$	column matrix of independent variable
u	reduced latitude
V	arbitrary location of space vehicle
$W = \sqrt{1 - e^2 \sin^2 \phi}$	
X,Y,Z	rectangular cartesian coordinate system (to be defined)
x,y,z	coordinates of point in X,Y,Z coordinate system
α	earth subtense angle
β	azimuth of point on earth's visible horizon as seen from vehicle
γ	elevation of vehicle with respect to spheroid
γ'	elevation of vehicle with respect to sphere

Δ	small finite change in variable
δN	height of spheroid above geoid
$\delta\phi$	difference between geocentric and geodetic latitudes
$\{E\}, \{v\}$	column matrices of residuals and corrections, respectively
ϵ	element of E
η, ξ	components of deflection of the vertical
θ	angle between geocentric radii of vehicle and radar station
λ	longitude
μ	azimuth of plane of the vertical
ν	deflection of the vertical
ρ, τ	unit vectors
σ	standard deviation
ϕ	geodetic latitude
ϕ'	geocentric latitude
ψ	polar angle
ω	longitude east of origin of datum, $\lambda_s - \lambda_0$

Subscripts:

H	horizon
I	instrument
i	particular azimuth or point on earth's visible horizon
k	number of observation equations
m	type of observation
n	particular position of vehicle
o	variable referenced to origin of spheroid
s	radar-station-referenced variables (except where defined differently)
T	total

sv measurement from radar station to vehicle
true true location or value
v vehicle-referenced variables

Notations:

$\{ \}$ column matrix

$[]$ square matrix

$[\]$ row matrix

$\| \|$ rectangular matrix

$*$ least-squares estimate

Matrix exponents:

T indicates the transpose

-1 indicates the inverse

A bar over a symbol (or group of symbols) indicates a vector.

An arc over a symbol (or group of symbols) indicates an arc measure.

A primed function indicates a partial derivative.

BRIEF REVIEW OF GEODETIC THEORY

As is well known, the earth's figure is an irregular one. Furthermore, the density varies from region to region. Therefore, in order to locate the various features on the earth's solid surface, geodesy relies on two reference surfaces, the geoid and the oblate ellipsoid. The geoid is defined most precisely as the equipotential surface of the earth's gravitation and rotation which in the open ocean will coincide on the average with the mean sea level and to which the waters on the ocean would tend to conform if allowed to flow into very narrow and shallow canals cut through the land (refs. 1 and 2). Thus, the geoid coincides, on the average, with the mean sea level in the open seas but will rise under the land masses in proportion to the attraction of the land mass above mean sea level.

If the earth's surface were a perfect ellipsoid, the geoid would almost be an exact ellipsoid. However, because of the irregularities in the earth's

shape and the variations in its internal density, the geoid, although smooth, is an irregular surface and follows the earth's mean contour. According to reference 1 the geoid departs from the spheroidal shape by a few hundred feet and in inclination by as much as a minute of arc. (Although there is a distinction between the terms spheroid and ellipsoid, the two will be considered synonymous in this paper.)

The vertical axis of a level theodolite at sea level is coincident with the direction of gravity at the point of observation. Further, the horizontal axis of a level spirit level at sea level is normal to the direction of gravity at the point of observation. Therefore, since the direction of gravity at sea level is everywhere normal to the geoid, this is the surface to which the geodesist references all of his measurements. As a result, observables such as astronomic latitude, longitude, and elevation above sea level are defined in terms of the geoid.

Astronomic latitude and longitude in the conventional sense are defined as follows: the astronomical latitude is the angle between the tangent to the observer's plumb line at its intersection with the geoid and the earth's mean equatorial plane and is positive north. The astronomical longitude may be defined as the angle between the mean meridian plane of Greenwich (this plane contains the normal to the geoid at Greenwich and is parallel to the earth's mean rotational axis) and the plane normal to the earth's mean equatorial plane and containing the observer's geoidal normal mentioned and is measured positive east of Greenwich. To be precise in these definitions, it would be necessary to substitute the word "instantaneous" for the word "mean." However, since the orientation of the reference ellipsoid relative to the geoid must fulfill the condition that its minor axis be parallel to the earth's mean axis of rotation, the earth's spin axis will be considered to be its mean axis of rotation and only the conventional definitions of astronomic latitude and longitude will be considered. (See ref. 3.)

The elevation of a point is defined as its height above the geoid. For the purposes of this paper, this definition will suffice. The act of measuring an elevation by means of a spirit level assumes that an equipotential surface adjacent to the geoid is everywhere equidistant from the geoid. However, this is not true and, hence, heights obtained by spirit leveling over great distances must be corrected. (See refs. 1, 2, and 4.)

An oblate spheroid or ellipsoid of revolution approximates the geoid very closely and is the second surface with which the geodesist is concerned. The spheroid is a mathematical surface and is by definition smooth. It is, therefore, the ideal surface for geodetic computations and for locating geodetic points.

The geodetic latitude is the angle between the earth's equatorial plane and the normal to the ellipsoid containing the observer's projected position on the geoid. The geodetic longitude is the angle between the meridian plane containing the same ellipsoidal normal and the meridian plane of Greenwich where both planes are parallel to the earth's mean axis of rotation. (See ref. 3.)

If the astronomical latitude and longitude of an arbitrary point and the azimuth of an additional point are desired, it is required only to observe these quantities. However, if it is desired to locate a second point relative to the first, it is, for the most accurate determination, necessary to connect the two points by first-order triangulation; select an ellipsoid which will best fit the region surveyed and define its orientation with respect to the first point; then, calculate the geodetic coordinates of the second point on the assumed ellipsoid. The resulting geodetic coordinates of the second point would, most probably, not agree with those determined astronomically. The major reason for this discrepancy is the varying density of the earth from point to point on its surface, which would cause the vertical (or direction of gravity) at the point to have a direction different from that of the normal to the ellipsoid at that point.

The angle between the vertical and the normal to the ellipsoid is defined as the deflection of the vertical. The deflection of the vertical can be resolved into two components ξ and η , which are conventionally positive if the inward vertical is directed to the south or west of the normal. (See ref. 1.) The component of deviation in the meridian plane η is given by the relation

$$\eta = \text{Astrolatitude} - \text{Geodetic latitude} \quad (1)$$

The component in the prime vertical plane ξ is given by the relation

$$\xi = (\text{Astrolongitude} - \text{Geodetic longitude}) \cos \phi \quad (2)$$

In some texts the definitions of ξ and η are reversed.

A second reason for the discrepancies between astronomic and geodetic coordinates may be a poorly defined datum or spheroid. The definition of the datum of a triangulation network requires eight constants. First, the minor axis of the ellipsoid is always defined to be parallel to the earth's spin axis, and thereby, the orientation of the triangulation on the ellipsoid can be determined through the Laplace equation. This involves first the observation of the three constants, the astronomic latitude, longitude, and azimuth and then the assigning of values to the two components of the deflection of the vertical ξ_0 and η_0 . Secondly, values must be assigned to the two ellipsoidal parameters a and f . Finally, the height of the ellipsoid above the geoid δN_0 is defined to complete the definition of a datum.

The definition of a datum, except for the astronomic latitude, longitude, and azimuth, is purely arbitrary. Generally, the geodetic and astronomic coordinates may be defined as being equal at the origin. However, such an assumption may not yield the best fit to the geoid. Consequently, it would be desirable, in such cases, to define ξ and η to be different from zero.

The last three constants ξ_0 , η_0 , and δN_0 represent an attempt to define the center of the ellipsoid which, ideally, would be at the earth's center of gravity. Because of the irregular shape of the geoid and the uncertainty in the magnitudes of a and f which best fit the geoid as a whole,

this does not hold. Consequently, many independent surveys have been carried out which have resulted in several spheroids (European, American, etc.), each defined so as to fit best the particular area in question. Attempts to connect the various spheroids have shown serious discrepancies. The failure to connect the various spheroids is the result of two factors: the true deflection of the vertical at the origin for each datum is unknown, and the constants of the best fitting world ellipsoid are unknown.

Stokes has described and developed the theory of a method which, if gravity observations over all the earth were available, would permit the calculation of geoid undulations at any desired point on the earth's surface. Vening Meinesz later extended the theory to show that the components of the deflection of the vertical could also be determined from gravity observations. (See ref. 5.) When the earth as a whole is considered, the holdings of gravimetric data are not by any means complete although enough data exist to make a few very good approximations. In the southern hemisphere, observations are sparse indeed. In the northern hemisphere, the picture is much brighter but still sadly lacking.

The job of collecting gravimetric data is a gigantic and expensive undertaking. Therefore, a more efficient and direct method of determining the best fitting world ellipsoid is desirable.

There are several ways to determine the earth's figure. The most frequently used approach has been to observe the differences in astronomic latitude and longitude from place to place and to connect the points of observation by triangulation so as to determine the lengths of the degree of latitude and/or longitude in different parts of the earth and to deduce the earth's figure from the results. Another approach has been through measurement of the variation of gravity between the equator and poles of the earth. It was the observation of the direction and intensity of the earth's gravitation which led to the discovery of irregularities in the earth's figure and internal structure.

There have been a number of attempts to derive an ellipsoid which will best fit (as a whole) the earth's figure. Until 1959, the international ellipsoid ($a = 6\,378\,388$ meters; $f = 1/297$) was generally accepted as the truest representation of the earth's figure. However, in reference 6, observations of the Vanguard satellite were shown to indicate a flattening of $1/298.3$. These findings resulted in a rash of additional derivations of the earth's ellipsoidal parameters. However, there is still disagreement.

Once the parameters of the best fitting ellipsoid have been established it would be necessary either to change the spheroids of the various geodetic systems and recompute the station coordinates or to derive formulas for converting from the old datum to the new. The two alternatives differ only in the extent to which the computations are carried. For the first alternative the coordinates of all geodetically established points would be changed. For the second, a convenient formula would be provided for converting the coordinates of any desired point on the old spheroid to those of the new.

If it is desired to change a spheroid, one approach would be first to redefine a , f , ξ_0 , η_0 , and δN_0 and then compute the coordinates of the observed station of the triangulation network again. There are available several methods, of which the methods of Vening Meinesz and de Graaff-Hunter are two that allow for a change in the datum without going through the rigorous computation of the triangulation just outlined (refs. 1 and 7). Essentially, if $\Delta\eta_0$, $\Delta\xi_0$, $\Delta\delta N_0$, Δa , and Δf are the changes to be added to η_0 , ξ_0 , δN_0 , a , and f , respectively, the corresponding change in the geodetic latitude $\Delta\phi_s$, longitude $\Delta\lambda_s$, and station height Δh_s may be obtained directly. The de Graaff-Hunter method was chosen for this study. In functional notation

$$\left. \begin{aligned} \Delta\phi_s &= L(\Delta a, \Delta f, \Delta\xi_0, \Delta\eta_0, \Delta\delta N_0) \\ \Delta\lambda_s &= M(\Delta a, \Delta f, \Delta\xi_0, \Delta\eta_0, \Delta\delta N_0) \\ \Delta h_s &= S(\Delta a, \Delta f, \Delta\xi_0, \Delta\eta_0, \Delta\delta N_0) \end{aligned} \right\} \quad (3)$$

Assuming the availability of all desirable data, to establish a truly world geodetic system and relate all existing datums to it by the previously discussed procedures is an overwhelming task. Therefore, a method which would allow the datum to be adjusted in addition to deriving a truly earth-centered, best-fitting world ellipsoid would be of great importance. The development of such a method is the purpose of this paper. Because of the magnitude of the problem, however, the present report proposes to develop the concept according to certain assumptions and apply it to a particular case. Some of the associated problems and areas for additional work are pointed out as well as indications of the merit of the method.

ASSUMPTIONS AND GENERAL THEORY

If an artificial earth satellite is assumed to be at a point V_n in space and $R_{v,n}$ units from the earth's center at time t_0 , as is shown in figure 1, and, by some suitable optical means, a point T_1 , on the earth's visible horizon, is observed, the vector $\overline{V_n T_1}$ (assuming no atmosphere) is the line of sight and is tangent to the earth at T_1 . If the earth's figure is assumed to be a smooth oblate ellipsoid, the vector $\overline{V_n T_1}$ is perpendicular to the normal N_1 to the ellipsoid at T_1 . The angle $\alpha_{n,i}$ included by vectors $\overline{V_n T_1}$ and $\overline{R_{v,n}}$ in the plane of observation is defined as the T^{th} subtense angle of the earth.

It is obvious that, in addition to being dependent on the size and shape of the earth, $\alpha_{n,i}$ will vary in magnitude as the plane containing $\overline{V_n T_1}$ and $\overline{R_{v,n}}$ rotates about $R_{v,n}$ and as the location of V_n changes with respect to the earth's center. Further, if the earth is assumed to be a smooth oblate ellipsoid of revolution, due to the symmetry of the ellipsoid, $\alpha_{n,i}$ is

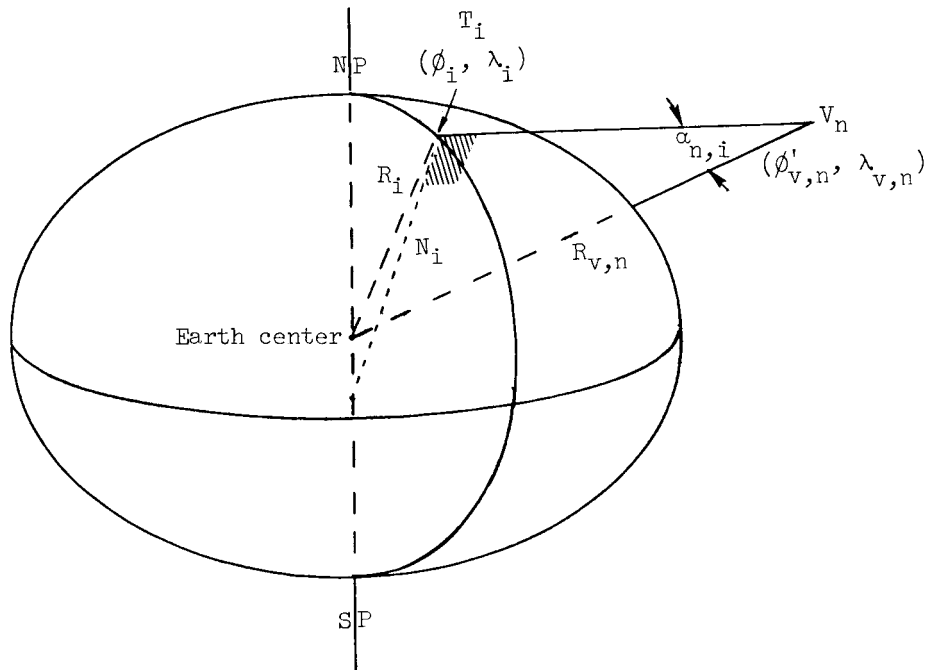


Figure 1.- Optical observation of a point on earth's visible horizon from a satellite.

independent of the vehicle's longitude $\lambda_{v,n}$. Then, if β_i is the azimuth of the plane formed by $\overline{V_n T_i}$ and $\overline{R_{v,n}}$ measured in a plane normal to $\overline{R_{v,n}}$ and clockwise from the meridian plane of V_n and if the spherical coordinates of V_n are $R_{v,n}$, $\phi'_{v,n}$, $\lambda'_{v,n}$, where $\phi'_{v,n}$ is the geocentric latitude of V_n , the earth subtense angle for the i^{th} point of tangency and the n^{th} position in space is written as

$$\alpha_{n,i} = F(a, f, R_{v,n}, \phi'_{v,n}, \beta_i) \quad (4)$$

In this analysis, the assumption is made that the lines of sight are truly tangent to the solid surface of the earth and are not intercepted by clouds, haze, or atmospheric scattering. Effects of atmospheric refraction are neglected.

The earth's solid surface is an irregular figure. However, if it is assumed that the elevations of various features on the earth's surface vary in a random manner about some mean elevation, the summation of all the elevations about that mean over any path would be zero. That is, if the earth's mean shape is assumed to be that of an oblate ellipsoid of revolution, there is an ellipsoid with semimajor axis a and flattening f . If this ellipsoid is placed so that its geometric center is coincident with that of the earth (the earth's geometric center and center of gravity will be taken to be synonymous), the sum of all the elevations about it will be zero. Thus, if a large number of subtense angles are observed from an artificial satellite at different

positions in space and the geocentric radii and latitudes are observed from some earth-based observation station, the datum to which the observation station is referenced could be improved by the method of least squares.

An ellipsoid determined in the manner described here will represent the earth's mean geometrical shape. Further, the resulting fit to the mean geometrical shape will be a function of both the number of observations and the particular surface features observed. For example, if observations are made predominately in a mountainous region, the resulting ellipsoid would be larger than desired. On the other hand, however, if observations are made only to the surface of the oceans, the resulting ellipsoid would be smaller than desired. (See previous section.) Thus, if observations should be made to all regions of the earth's surface, the ratio of the number of observations on a particular feature, such as mountain ranges and oceans, to the total can be assumed to be proportional to the ratio of the area of the feature to the total surface area of the earth, and the resulting ellipsoid would represent a best fit to the earth's mean geometric shape. An error exists in using horizon measurements in mountainous regions, because of the fact that the lines of sight from a satellite to the horizon may be unable to reach the ground in valleys. Because the oceans and seas cover approximately 2/3 of the earth's surface and the mountainous regions comprise a very small portion of the total surface area, this error would probably influence the resulting ellipsoid only slightly.

The two spherical coordinates of the vehicle $R_{v,n}$ and $\phi'_{v,n}$ in equation (4) are dependent on the geodetic coordinates of the observation point and the observations necessary to fix the location of the vehicle in space. In functional notation

$$\left. \begin{aligned} \phi'_{v,n} &= \Phi(a, f, \phi_s, \lambda_s, h_s, o_{m,n}, o_{m-1,n}, \dots, o_{1,n}) \\ \lambda_{v,n} &= \Lambda(a, f, \phi_s, \lambda_s, h_s, o_{m,n}, o_{m-1,n}, \dots, o_{1,n}) \\ R_{v,n} &= P(a, f, \phi_s, \lambda_s, h_s, o_{m,n}, o_{m-1,n}, \dots, o_{1,n}) \end{aligned} \right\} \quad (5)$$

where the values of m ($m = 1, 2, 3, \dots$) represent the particular observations necessary for the purpose of fixing the position of the vehicle in space.

As explained in the section on geodetic theory, the geodetic coordinates of the observation station in equation (5) are not independent of the origin of the datum. Thus, since the datum is arbitrarily defined so as to obtain a best fit of the ellipsoid to the surveyed area, the spherical coordinates of the vehicle must be considered to be in error. The error in each is given by its total differential. Consequently, the error equations are

$$\left. \begin{aligned} d\phi'_{v,n} &= \frac{\partial\phi}{\partial a} da + \frac{\partial\phi}{\partial f} df + \frac{\partial\phi}{\partial\phi_s} d\phi_s + \frac{\partial\phi}{\partial\lambda_s} d\lambda_s + \frac{\partial\phi}{\partial h_s} dh_s + \sum_{m=1}^p \frac{\partial\phi}{\partial o_{m,n}} do_{m,n} \\ d\lambda'_{v,n} &= \frac{\partial\lambda}{\partial a} da + \frac{\partial\lambda}{\partial f} df + \frac{\partial\lambda}{\partial\phi_s} d\phi_s + \frac{\partial\lambda}{\partial\lambda_s} d\lambda_s + \frac{\partial\lambda}{\partial h_s} dh_s + \sum_{m=1}^p \frac{\partial\lambda}{\partial o_{m,n}} do_{m,n} \\ dR'_{v,n} &= \frac{\partial P}{\partial a} da + \frac{\partial P}{\partial f} df + \frac{\partial P}{\partial\phi_s} d\phi_s + \frac{\partial P}{\partial\lambda_s} d\lambda_s + \frac{\partial P}{\partial h_s} dh_s + \sum_{m=1}^p \frac{\partial P}{\partial o_{m,n}} do_{m,n} \end{aligned} \right\} \quad (6)$$

Those variables $o_{m,n}$ which are to be observed by some suitable method may change with the ellipsoidal parameters a and f and the geodetic coordinates of the observation station. Thus, if it is assumed that the error in each $do_{m,n}$ is given by the relation

$$do_{m,n} = \frac{\partial o_{m,n}}{\partial a} da + \frac{\partial o_{m,n}}{\partial f} df + \frac{\partial o_{m,n}}{\partial\phi_s} d\phi_s + \frac{\partial o_{m,n}}{\partial\lambda_s} d\lambda_s + \frac{\partial o_{m,n}}{\partial h_s} dh_s \quad (7)$$

upon substituting equation (7) into equation (6) and collecting coefficients of like terms, error equations for the spherical coordinates of the vehicle as functions of the errors in the ellipsoidal parameters and geodetic coordinates and height of the observation station are

$$\left. \begin{aligned} d\phi'_{v,n} &= \Phi'_{a,n} da + \Phi'_{f,n} df + \Phi'_{\phi_s,n} d\phi_s + \Phi'_{\lambda_s,n} d\lambda_s + \Phi'_{h_s,n} dh_s \\ d\lambda'_{v,n} &= \Lambda'_{a,n} da + \Lambda'_{f,n} df + \Lambda'_{\phi_s,n} d\phi_s + \Lambda'_{\lambda_s,n} d\lambda_s + \Lambda'_{h_s,n} dh_s \\ dR'_{v,n} &= P'_{a,n} da + P'_{f,n} df + P'_{\phi_s,n} d\phi_s + P'_{\lambda_s,n} d\lambda_s + P'_{h_s,n} dh_s \end{aligned} \right\} \quad (8)$$

where the Φ' , Λ' , and P' coefficients represent the sums of the partial derivatives of the particular function with respect to the variable indexed.

The differential operator d of equation (8) is equivalent to the Δ in the relations of equation (3). The mathematical difference in the two, for example, is that da represents an infinitesimal error in the variable whereas Δa represents a small but finite error in the variable. However, it is assumed that the coefficients of the unknowns are linear over a finite range of the variables. Thus, the two mathematical representations are taken to be equal, and, therefore, the error equations for the spherical coordinates of the vehicle can be determined as functions of a change in the datum.

The relations of equation (3) are linear functions of their variables. Therefore,

$$\left. \begin{aligned} \Delta\phi_s &= L'_a\Delta a + L'_f\Delta f + L'_\eta\Delta\eta_0 + L'_\xi\Delta\xi_0 + L'_{\delta N}\Delta\delta N_0 \\ \Delta\lambda_s &= M'_a\Delta a + M'_f\Delta f + M'_\eta\Delta\eta_0 + M'_\xi\Delta\xi_0 + M'_{\delta N}\Delta\delta N_0 \\ \Delta h_s &= S'_a\Delta a + S'_f\Delta f + S'_\eta\Delta\eta_0 + S'_\xi\Delta\xi_0 + S'_{\delta N}\Delta\delta N_0 \end{aligned} \right\} \quad (9)$$

where primed-function names represent the coefficient of the corresponding unknown, which is given as a subscript. Then, after substituting equation (9) into equation (8),

$$\left. \begin{aligned} \Delta\phi'_{v,n} &= \left(\phi'_{a,n} + L'_a\phi'_{s,n} + M'_a\phi'_{\lambda_s,n} + S'_a\phi'_{h_s,n} \right) \Delta a + \left(\phi'_{f,n} + L'_f\phi'_{s,n} + M'_f\phi'_{\lambda_s,n} \right. \\ &\quad \left. + S'_f\phi'_{h_s,n} \right) \Delta f + \left(L'_\eta\phi'_{s,n} + M'_\eta\phi'_{\lambda_s,n} + S'_\eta\phi'_{h_s,n} \right) \Delta\eta_0 + \left(L'_\xi\phi'_{s,n} + M'_\xi\phi'_{\lambda_s,n} \right. \\ &\quad \left. + S'_\xi\phi'_{h_s,n} \right) \Delta\xi_0 + \left(L'_{\delta N}\phi'_{s,n} + M'_{\delta N}\phi'_{\lambda_s,n} + S'_{\delta N}\phi'_{h_s,n} \right) \Delta\delta N_0 \\ \Delta\lambda'_{v,n} &= \left(\lambda'_{a,n} + L'_a\lambda'_{s,n} + M'_a\lambda'_{\lambda_s,n} + S'_a\lambda'_{h_s,n} \right) \Delta a + \left(\lambda'_{f,n} + L'_f\lambda'_{s,n} + M'_f\lambda'_{\lambda_s,n} \right. \\ &\quad \left. + S'_f\lambda'_{h_s,n} \right) \Delta f + \left(L'_\eta\lambda'_{s,n} + M'_\eta\lambda'_{\lambda_s,n} + S'_\eta\lambda'_{h_s,n} \right) \Delta\eta_0 + \left(L'_\xi\lambda'_{s,n} + M'_\xi\lambda'_{\lambda_s,n} \right. \\ &\quad \left. + S'_\xi\lambda'_{h_s,n} \right) \Delta\xi_0 + \left(L'_{\delta N}\lambda'_{s,n} + M'_{\delta N}\lambda'_{\lambda_s,n} + S'_{\delta N}\lambda'_{h_s,n} \right) \Delta\delta N_0 \\ \Delta R'_{v,n} &= \left(P'_{a,n} + L'_aP'_{s,n} + M'_aP'_{\lambda_s,n} + S'_aP'_{h_s,n} \right) \Delta a + \left(P'_{f,n} + L'_fP'_{s,n} + M'_fP'_{\lambda_s,n} \right. \\ &\quad \left. + S'_fP'_{h_s,n} \right) \Delta f + \left(L'_\eta P'_{s,n} + M'_\eta P'_{\lambda_s,n} + S'_\eta P'_{h_s,n} \right) \Delta\eta_0 + \left(L'_\xi P'_{s,n} + M'_\xi P'_{\lambda_s,n} \right. \\ &\quad \left. + S'_\xi P'_{h_s,n} \right) \Delta\xi_0 + \left(L'_{\delta N}P'_{s,n} + M'_{\delta N}P'_{\lambda_s,n} + S'_{\delta N}P'_{h_s,n} \right) \Delta\delta N_0 \end{aligned} \right\} \quad (10)$$

If the coefficients of equation (8) can be determined and are linear in the corresponding variables, the change in the spherical coordinates of the vehicle due to a change in the datum can be computed by equation (10). Thus, only the error equation for $\alpha_{n,i}$ remains to be found.

Since the function F of equation (4) may not be linear in its variables, it is linearized by expanding it into a Taylor series and ignoring all terms containing powers of the variables higher than one. Thus, if

$$d\beta_i = 0$$

for all the remaining variables of F

$$\alpha_{n,i} = (F)_{n,i} + \left(\frac{\partial F}{\partial a}\right)_{n,i} da + \left(\frac{\partial F}{\partial f}\right)_{n,i} df + \left(\frac{\partial F}{\partial R_{v,n}}\right)_{n,i} dR_{v,n} + \left(\frac{\partial F}{\partial \phi'_{v,n}}\right)_{n,i} d\phi'_{v,n} \quad (11)$$

where a partial derivative of a function enclosed by parentheses and subscripted indicates that it is to be evaluated for a particular azimuth at a point.

Equation (11) can be interpreted as stating that

$$\begin{aligned} \Delta\alpha_{n,i} &= \text{Error in } \alpha_{n,i} = \alpha_{n,i} - (F)_{n,i} = \alpha_{n,i} (\text{true}) - \alpha_{n,i} (\text{approximate}) \\ &= \left(\frac{\partial F}{\partial a}\right)_{n,i} da + \left(\frac{\partial F}{\partial f}\right)_{n,i} df + \left(\frac{\partial F}{\partial R_{v,n}}\right)_{n,i} dR_{v,n} + \left(\frac{\partial F}{\partial \phi'_{v,n}}\right)_{n,i} d\phi'_{v,n} \quad (12) \end{aligned}$$

Substituting the necessary expressions from equation (10) into equation (12) and collecting the coefficients of like terms gives the following error equation for $\alpha_{n,i}$:

$$\Delta\alpha_{n,i} = (F'_a)_{n,i} da + (F'_f)_{n,i} df + (F'_{\eta_0})_{n,i} d\eta_0 + (F'_{\xi_0})_{n,i} d\xi_0 + (F'_{\delta N_0})_{n,i} d\delta N_0 \quad (13)$$

where F' is the coefficient for the variable subscripted.

Generalizing further, let

$$\left. \begin{aligned} \epsilon_k &= \Delta\alpha_{n,i} \\ [q_k] &= (F'_a)_{n,i}, (F'_f)_{n,i}, (F'_{\eta_0})_{n,i}, (F'_{\xi_0})_{n,i}, (F'_{\delta N_0})_{n,i} \\ \{U\} &= da, df, d\eta_0, d\xi_0, d\delta N_0 \end{aligned} \right\} \quad (14)$$

where

$$k = 1, 2, \dots$$

is a particular observation. Then, in matrix notation

$$\epsilon_k = [q_k]\{U\} \quad (15)$$

Equation (15) represents only one observation. For a least-squares determination of the unknowns to be valid, it is necessary to have a large number of observations. In the theory presented so far, two varying indices i and n have been used. Thus, if $\{E\}$ is chosen to represent a column matrix of k elements and $\|Q\|$ is chosen to represent a k by j rectangular matrix of the coefficients, where j is the number of unknowns and v is a column matrix of corrections, equation (15) can be rewritten to read

$$\|Q\|\{U\} = \{E\} + \{v\} \quad (16)$$

which after requiring the sum of the squares of v to be a minimum becomes

$$\|Q\|^T\|Q\|\{U^*\} = \|Q\|^T\{E\} \quad (17)$$

The product $\|Q\|^T\|Q\|$ is a square matrix and can be inverted. Therefore,

$$\{U^*\} = \left[\|Q\|^T\|Q\| \right]^{-1} \|Q\|^T\{E\} \quad (18)$$

is the matrix representation of the least-squares solution for the unknowns (ref. 8). A similar solution would be obtained for weighted observations.

As noted previously, atmosphere refraction has been neglected in development of the theory. If equation (4) is referred to, it will be seen that the observed subtense angle can easily be corrected for atmospheric refraction provided a suitable correction is available.

THE EXPRESSIONS FOR α AND ITS ERROR EQUATION

The theory given in the previous section was very general in that functional notation alone was used. In this section the expressions for $\alpha_{n,i}$ and its error equation are given.

The Expression for α

If a satellite is located at a point V_n in space and if by some suitable optical means (assuming no atmospheric refraction) a point T_i on the visible horizon is observed, the subtense angle $\alpha_{n,i}$ is given by the functional relation of equation (4). Furthermore, if the earth is assumed to be a smooth

oblate ellipsoid, the line of sight $\overline{V_n T_i}$ is tangent to the earth at T_i and perpendicular to the normal N_i . Then, it is shown in appendix A that the desired expression for $\alpha_{n,i}$ is

$$\alpha_{n,i} = \tan^{-1} \left(\frac{-B_{n,i} + \sqrt{B_{n,i}^2 - 4A_{n,i}C_{n,i}}}{2A_{n,i}} \right) \quad (19)$$

where

$$\left. \begin{aligned} A_{n,i} &= \left(\frac{R_{v,n}^2 - a^2}{R_{v,n}} \right)^2 (1-f)^4 + \left[R_{v,n}^2 (2f-f^2) \sin^2 \phi'_{v,n} + 2(R_{v,n}^2 - a^2)(1-f)^2 \right] (2f-f^2) \sin^2 \phi'_{v,n} + \left[(2f-f^2) \sin^2 \phi'_{v,n} + (1-f)^2 \left(1 - \frac{a^2}{R_{v,n}^2} \right) \right] (R_{v,n}^2 - a^2) (2f-f^2) \cos^2 \phi'_{v,n} \cos^2 \beta_i \\ B_{n,i} &= \left[(2f-f^2) \sin^2 \phi'_v + \left(1 - \frac{a^2}{R_{v,n}^2} \right) (1-f)^2 \right] a^2 (2f-f^2) \sin 2\phi'_{v,n} \cos \beta_i \\ C_{n,i} &= -a^2 \left[\left(1 - \frac{a^2}{R_{v,n}^2} \right) (1-f)^4 + \left(2 - \frac{a^2}{R_{v,n}^2} \right) (1-f)^2 (2f-f^2) \sin^2 \phi'_{v,n} + (2f-f^2)^2 \sin^4 \phi'_{v,n} \right] \end{aligned} \right\} \quad (20)$$

The positive sign is chosen so that $\alpha_{n,i}$ is a minimum when $\beta_{n,i}$ is zero.

The Error Equation for α

Before Φ , Λ , and P can be derived, it is necessary to decide what method will be used to fix the position of the satellite. Because of the high accuracy obtainable, radar was chosen for this paper. Further, it is assumed that perfect measurements are made.

There are several techniques for fixing the instantaneous position of a space vehicle by radar observations in use today. Since it is impractical to discuss each technique and its individual application to the theory of this work, the conventional measurements of range $R_{sv,n}$, azimuth $Az_{g,n}$, and the elevation above the horizon γ_n are taken to be observables.

The azimuth and elevation are geodetic azimuth and geodetic elevation. Thus, if the datum should be changed, $Az_{g,n}$ and γ_n must necessarily change.

On the other hand, however, the range $R_{sv,n}$ is a pure measurement which is solely a function of the radar accuracy. As a result, derivatives of $R_{sv,n}$ are taken to be zero.

With the assumptions stated herein in mind, it is shown in appendix B that the coefficients of the expressions in equation (8) can be expressed as follows: For $d\phi'_{v,n}$

$$\left. \begin{aligned} \phi'_{a,n} &= \frac{\partial \phi'_{v,n}}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \frac{\partial N_s}{\partial a} + \frac{\partial \phi'_{v,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial a} + \frac{\partial \phi'_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial a} \\ \phi'_{f,n} &= \frac{\partial \phi'_{v,n}}{\partial \phi'_s} \left(\frac{\partial \phi'_s}{\partial N_s} \frac{\partial N_s}{\partial f} + \frac{\partial \phi'_s}{\partial f} \right) + \frac{\partial \phi'_{v,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial f} + \frac{\partial \phi'_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial f} \\ \phi'_{\phi_s,n} &= \frac{\partial \phi'_{v,n}}{\partial \phi'_s} \left(\frac{\partial \phi'_s}{\partial N_s} \frac{\partial N_s}{\partial \phi_s} + \frac{\partial \phi'_s}{\partial \phi_s} \right) + \frac{\partial \phi'_{v,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \phi_s} + \frac{\partial \phi'_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \phi_s} \\ \phi'_{\lambda_s,n} &= \frac{\partial \phi'_{v,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \lambda_s} + \frac{\partial \phi'_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \lambda_s} \\ \phi'_{h_s,n} &= \frac{\partial \phi'_{v,n}}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial h_s} + \frac{\partial \phi'_{v,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial h_s} + \frac{\partial \phi'_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial h_s} \end{aligned} \right\} \quad (21)$$

For $d\lambda_{v,n}$

$$\begin{aligned}
 \Lambda'_{a,n} &= \Phi'_{a,n} \frac{\partial \Delta \lambda_{sv,n}}{\partial \phi'_{v,n}} + \frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial a} + \frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial a} \\
 \Lambda'_{f,n} &= \Phi'_{f,n} \frac{\partial \Delta \lambda_{sv,n}}{\partial \phi'_{v,n}} + \frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial f} + \frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial f} \\
 \Lambda'_{\phi_s,n} &= \Phi'_{\phi_s,n} \frac{\partial \Delta \lambda_{sv,n}}{\partial \phi'_{v,n}} + \frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \phi_s} + \frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \phi_s} \\
 \Lambda'_{\lambda_s,n} &= 1 + \Phi'_{\lambda_s,n} \frac{\partial \Delta \lambda_{sv,n}}{\partial \phi'_{v,n}} + \frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \lambda_s} + \frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \lambda_s} \\
 \Lambda'_{h_s,n} &= \Phi'_{h_s,n} \frac{\partial \Delta \lambda_{sv,n}}{\partial \phi'_{v,n}} + \frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial h_s} + \frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial h_s}
 \end{aligned} \tag{22}$$

For $dR_{v,n}$

$$\begin{aligned}
 P'_{a,n} &= \Phi'_{a,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{a,n} \frac{\partial R_{v,n}}{\partial \Delta \lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial a} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial a} \\
 P'_{f,n} &= \Phi'_{f,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{f,n} \frac{\partial R_{v,n}}{\partial \Delta \lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial f} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial f} \\
 P'_{\phi_s,n} &= \Phi'_{\phi_s,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{\phi_s,n} \frac{\partial R_{v,n}}{\partial \Delta \lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial \phi_s} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \phi_s} \\
 P'_{\lambda_s,n} &= \Phi'_{\lambda_s,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + (\Lambda'_{\lambda_s,n} - 1) \frac{\partial R_{v,n}}{\partial \Delta \lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial \lambda_s} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \lambda_s} \\
 P'_{h_s,n} &= \Phi'_{h_s,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{h_s,n} \frac{\partial R_{v,n}}{\partial \Delta \lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial h_s} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial h_s}
 \end{aligned} \tag{23}$$

Further, the coefficients of equation (12) are shown to be given by

$$\left. \begin{aligned} \left(\frac{\partial F}{\partial a} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial a} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial a} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial a} \\ \left(\frac{\partial F}{\partial f} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial f} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial f} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial f} \\ \left(\frac{\partial F}{\partial \phi'_{v,n}} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial \phi'_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial \phi'_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial \phi'_{v,n}} \\ \left(\frac{\partial F}{\partial R_{v,n}} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial R_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial R_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial R_{v,n}} \end{aligned} \right\} \quad (24)$$

The error equation for $\alpha_{n,i}$ is given in a general form by equation (13). As stated previously, the coefficients of equation (13) are functions of those of equations (8), (9), and (12).

For brevity, in this section the coefficients of equations (8) and (12) were given in equations (21), (22), (23), and (24) as functions of indicated partial derivatives. The indicated operations have been executed and the results are given in appendix C. To obtain the algebraic expressions for equations (21), (22), (23), and (24), it is required only to make necessary substitutions.

To complete the list of expressions for the solution of equation (13), it is only necessary to rearrange the equations due to de Graaff-Hunter to the form given by equation (9). This has been done and the results are given in appendix D.

The foregoing has shown that the error equation for $\alpha_{n,i}$ is determined in terms of quantities which are known, are assumed to be observed, or can be approximated. Thus, it is mathematically possible to derive by the method of least squares an earth-centered geodetic datum from earth subtense angles observed from an orbiting artificial earth satellite. It is now desirable to mention ways in which the theory can be applied and some of the conditions necessary for obtaining valid results. The former is the subject of the next section and the latter will be discussed in a subsequent section.

APPLICATION

There are several ways in which the theory of this paper can be applied, but each will have one point in common with the other. All methods will involve the direct observation of some angle either between two points on the earth's horizon, between a reference line and points on the horizon, between a reference star and points on the horizon, or between some other reference and points on the horizon. The approach chosen for a particular application will be largely dependent on the type of instruments available and their accuracies.

For comparison and discussion this section will consider two applications. One is simply that of observing the subtense angle directly and another is that of observing angles from a polar star to points on the earth's visible horizon.

There is no need to discuss the first approach in any greater detail. The assumption of being able to measure the earth's subtense angle by some means either directly or indirectly was the basis for the derivations which precede this section. The only drawback to such an approach is that the geocentric radius is not a physical reality, but mathematically defined. Therefore, instrumentation would be required to operate in such a manner as to allow the precise determination of the direction of the geocentric radius.

The approach of observing directly the angle between the direction of a polar star and the line of sight to a point on the earth's horizon is less complicated from the standpoint of instrumentation than that of observing subtense angles directly. Since the direction of a polar star would be known, it would only be required to observe the direction of a point on the horizon relative to the star. To utilize the "polar angle" determined in this manner, only slight modification of the procedure developed in this paper is necessary.

Consider figure 2 in which the Z' -axis is considered to pass through the vehicle's center and is positive in the direction of a polar star. If the star is considered to lie directly above the earth's north pole, the Z' -axis is always parallel to the earth's spin axis regardless of the location of the vehicle, since the star is essentially infinitely distant. As a result, Z' would translate about the earth's spin axis and, if $\psi_{n,i}$ is the polar angle,

$$\cos \psi_{n,i} = -(\sin \phi'_{v,n} \cos \alpha_{n,i} - \cos \phi'_{v,n} \sin \alpha_{n,i} \cos \beta_1) \quad (25)$$

Then,

$$d\psi_{n,i} = \frac{d\psi_{n,i}}{d\phi'_{v,n}} d\phi'_{v,n} + \frac{d\psi_{n,i}}{d\alpha_{n,i}} d\alpha_{n,i} \quad (26)$$

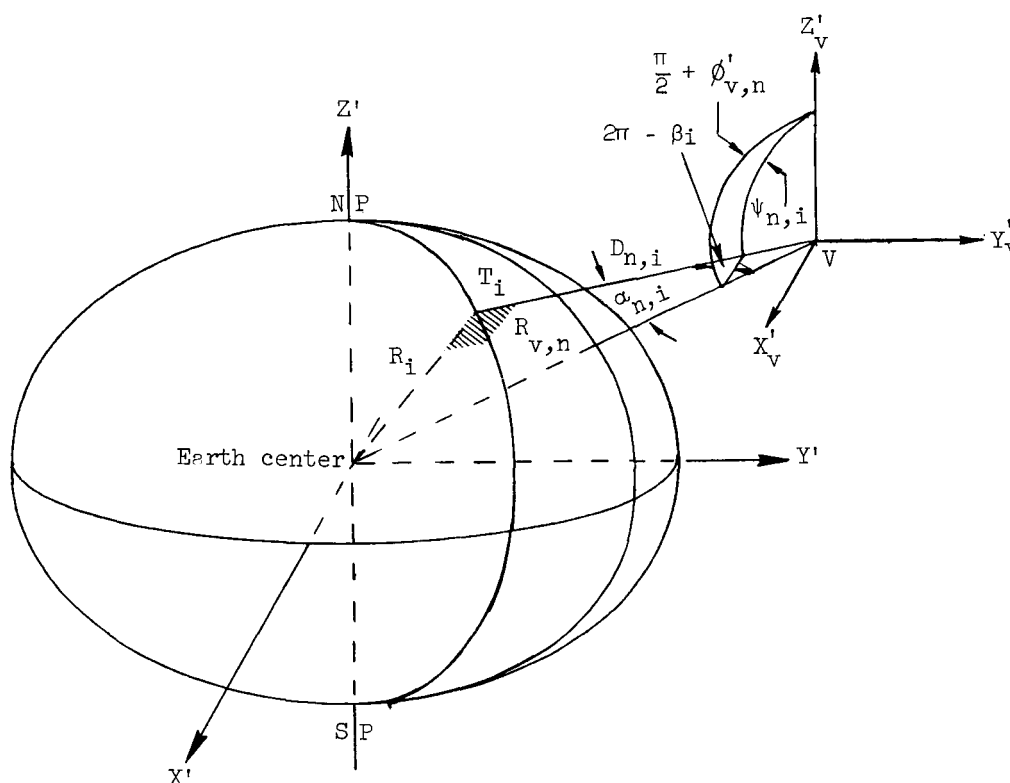


Figure 2.- Geometric definition of polar angle and its relation to subtense angle, azimuth of observation, and spherical latitude of a satellite.

where

$$\left. \begin{aligned} \frac{\partial \psi_{n,i}}{\partial \phi'_{v,n}} &= \frac{\cos \phi'_{v,n} \cos \alpha_{n,i} + \sin \phi'_{v,n} \sin \alpha_{n,i} \cos \beta_i}{\sin \psi_{n,i}} \\ \frac{\partial \psi_{n,i}}{\partial \alpha_{n,i}} &= - \frac{\sin \phi'_{v,n} \sin \alpha_{n,i} + \cos \phi'_{v,n} \cos \alpha_{n,i} \cos \beta_i}{\sin \psi_{n,i}} \end{aligned} \right\} \quad (27)$$

Therefore, since the expressions $d\phi'_{v,n}$ and $d\alpha_{n,i}$ are known and the partials of equation (25) are given by equation (27), after substitution the procedure becomes the same as that for the subtense angle.

In the absence of real data a computer program was written for the purpose of simulating data. The results were then used to determine the resulting earth-centered datum.

The method of simulating measurements was briefly this: A datum was defined at a selected latitude and longitude along with the coordinates of a

radar station; various positions of the vehicle were assumed in terms of radar measurements, and the approximate subtense angles were computed for certain values of β_1 ; errors were then assumed to exist in the datum, and the true coordinates of the radar station were determined by the method of de Graaff-Hunter (appendix D); finally, the assumed azimuth and elevation of the vehicle were corrected by equations (B14) and (B15) in appendix B, and the true subtense angles were found for the assumed values of β_1 . A similar procedure was used for the polar method.

The coordinates of the origin of the datum and the radar station and the errors in the datum are given in table I. For simplicity, the vehicle was assumed to be moving away from the earth at a constant elevation of approximately $57^{\circ}17'44.8''$ and azimuth of 225° . Observations were assumed to be initiated when

$$\beta_1 = 0.0^{\circ}$$

and

$$R_{SV} = 200\ 000\ \text{meters}$$

where β_1 was increased in steps according to the relation

$$\beta_{i+1} = \beta_i + \frac{\pi}{100}$$

whereas R_{SV} was increased in increments of 200 000 meters. One observation was assumed to be taken for every new value of β_1 and R_{SV} . In table II the resulting angles and residuals are given for the assumed true and approximate subtense and polar methods for each geocentric latitude and longitude. Fifty observations were taken and a solution was found after each, beginning with the fifth observation.

TABLE I.- DEFINED CONSTANTS FOR TEST CASE

Variable	ϕ , radians	λ , radians	h, meters	a, meters	f	ξ , radians	η , radians
Origin of datum	0.52359870	-1.3089967	10.0	6378388.0	336.70834×10^{-5}	-9.692736×10^{-6}	1.4515551×10^{-5}
Radar station	.58067831	-1.3835296	.0			2.4242184	2.0000000
Change in datum			-8.0	-223.0	-1.4673498	4.8484378	-1.2120342

TABLE II.- TRUE AND APPROXIMATE SUBTENSE AND POLAR ANGLES AND THE CORRESPONDING RESIDUALS FOR THE TEST CASE

$\phi_{v,n}^t$, radian	$R_{v,n}$, km	$\alpha_{n,i}$, radians			$\psi_{n,i}$, radians		
		Approximate	True	$\epsilon_{n,i}$	Approximate	True	$\epsilon_{n,i}$
0.56594264	6 541.37880	1.33965063	1.33965151	$0.08770755 \times 10^{-5}$	0.79708833	0.79711264	$0.24315266 \times 10^{-4}$
.55480848	6 712.47816	1.24795322	1.24795581	.25974485	.87816855	.87819012	.21578650
.54416163	6 885.13547	1.17941563	1.17941847	.28401545	.93747975	.93750010	.20355788
.53397802	7 059.23635	1.12312724	1.12312990	.26640128	.98578345	.98580305	.19593453
.52423424	7 234.67654	1.07481548	1.07481781	.23209582	1.02724484	1.02726388	.19040743
.51490762	7 411.36098	1.03226408	1.03226598	.19020476	1.06399216	1.06401077	.18605854
.50597644	7 589.20282	.99414064	.99414209	.14475538	1.09728131	1.09729955	.18246851
.49741994	7 768.12243	.95956622	.95956720	.09775206	1.12792277	1.12794071	.17941519
.48921833	7 948.04716	.92792249	.92792299	.05027029	1.15647264	1.15649031	.17676648
.48135288	8 128.91018	.89875341	.89875344	.00291677	1.18332969	1.18334714	.17443574
.47380581	8 310.65023	.87171028	.87170984	-.04395416	1.20878971	1.20880695	.17236107
.46656035	8 493.21103	.84651871	.84651781	-.09013019	1.23307787	1.23309492	.17049487
.45960066	8 676.54085	.82295781	.82295645	-.13548122	1.25636923	1.25638612	.16879836
.45291182	8 860.59189	.80084630	.80084450	-.17992547	1.27800223	1.27801896	.16723843
.44647975	9 045.32015	.78003308	.78003085	-.22340959	1.30048792	1.30050449	.16578592
.44029124	9 230.68488	.76039049	.76038783	-.26589677	1.32151635	1.32153280	.16441452
.43433382	9 416.64851	.74180942	.74180634	-.30735944	1.34196140	1.34197772	.16310018
.42859579	9 603.17636	.72419570	.72419222	-.34777188	1.36188406	1.36190024	.16182072
.42306611	9 790.23612	.70746735	.70746549	-.38712253	1.38135509	1.38135116	.16055561
.41773441	9 977.79799	.69155248	.69154823	-.42538246	1.40035701	1.40037294	.15928581
.41259096	10 165.83383	.67638758	.67638295	-.46253444	1.41898553	1.41900134	.15799372
.40762655	10 354.31826	.66191624	.66191126	-.49855760	1.43725084	1.43726650	.15666309
.40283257	10 543.22684	.64808811	.64808278	-.53343026	1.45517844	1.45519397	.15527905
.39820085	10 732.53739	.63485798	.63485231	-.56713004	1.47279003	1.47280541	.15382803
.39372376	10 922.22905	.62218514	.62217914	-.59963401	1.49010399	1.49011922	.15229783
.38939407	11 112.28216	.61003273	.61002643	-.63091897	1.50713594	1.50715101	.15067755
.38520498	11 302.67847	.59836731	.59836070	-.66096170	1.52389914	1.52391402	.14895761
.38115006	11 493.40105	.58715840	.58715151	-.68973931	1.54040478	1.54041949	.14712977
.37723235	11 684.43394	.57637817	.57637100	-.71722952	1.55666232	1.55667685	.14518707
.37341885	11 875.76199	.56600112	.56599369	-.74344100	1.57267968	1.57269400	.14312386
.36973146	12 067.37125	.55600384	.55599616	-.76826362	1.58846346	1.58847755	.14093579
.36615597	12 259.24861	.54636475	.54635683	-.79176875	1.60401905	1.60403290	.13861977
.36268757	12 451.38156	.53706396	.53705583	-.81390952	1.61935084	1.61936446	.13617394
.35932167	12 643.75854	.52808310	.52807475	-.83467103	1.63446228	1.63447565	.13359768
.35605398	12 836.36856	.51940512	.51939657	-.85404059	1.649356915	1.64936915	.13089154
.35288040	13 029.20127	.51101422	.51100550	-.87200791	1.66403410	1.66404691	.12805722
.34979706	13 222.24689	.50289571	.50288682	-.88885624	1.67849769	1.67851020	.12509756
.34680028	13 415.49623	.49503592	.49502688	-.90370756	1.69274753	1.69275974	.12201642
.34388659	13 608.94072	.48742210	.48741292	-.91743265	1.70678383	1.70679572	.11881872
.34105268	13 802.57201	.48004235	.48003306	-.92974426	1.72060630	1.72061785	.11551031
.33829541	13 996.38247	.47288556	.47287616	-.94063716	1.73421423	1.73422544	.11209795
.33561179	14 190.36460	.46594132	.46593182	-.95012715	1.74760655	1.74761741	.10858925
.33299901	14 384.51171	.45919986	.45919028	-.95822120	1.76078182	1.76079232	.10499261
.33045435	14 578.81689	.45265204	.45264240	-.96493241	1.77373831	1.77374844	.10131710
.32797527	14 773.27418	.44628929	.44627958	-.97027703	1.78647399	1.78648375	.09757244
.3255931	14 967.87751	.44010350	.44009376	-.97427444	1.79898660	1.79899597	.09376891
.32320416	15 162.62126	.43408711	.43407734	-.97694713	1.81127362	1.81128262	.08991726
.32090760	15 357.50020	.42823295	.42822316	-.97832064	1.82333237	1.82334097	.08602865
.31866752	15 552.50907	.42253428	.42252450	-.97842351	1.83515994	1.83516817	.08211453
.31648193	15 747.64323	.41698475	.41697498	-.97728717	1.84675533	1.84676115	.07818659

RESULTS AND DISCUSSION

In figure 3 the five arbitrarily defined constants of the datum are plotted as a function of the number of observations. The results for both the subtense-angle case and the polar-angle case are shown.

For both cases the results converge to the true values quickly, as is expected. Theoretically, since no random errors have been assumed to exist, convergence should be instantaneous. However, because of certain approximations and computer noise, instantaneous convergence will not result. Of the two methods, the polar-angle method appears to give the better results.

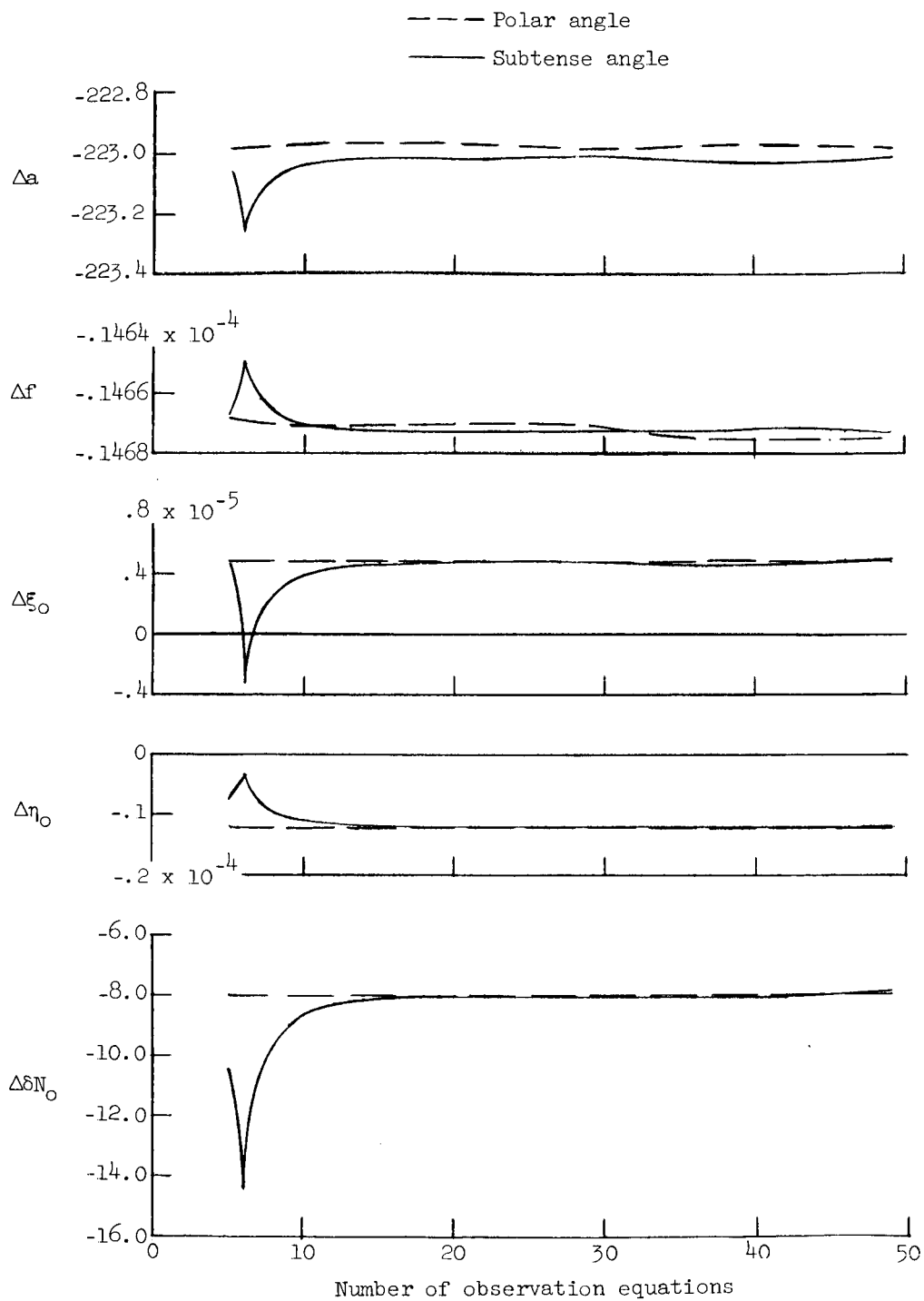


Figure 3.- Calculated change in datum for polar-angle and subtense-angle observations as a function of increasing observations for a test case.

One peculiarity is noticed in the case of the subtense angle. A sharp spike occurs in these curves at the sixth observation and is most pronounced in the $\Delta \xi_0$ curve. This oddity is not the result of computer noise.

There are two possible reasons for the spike effect. One is a poorly conditioned matrix, and the other is a possible breakdown in the equations at the sixth observation for this example of the subtense case.

A detailed study to determine which possibility, if either, is reasonable is beyond the scope of this paper. However, two cases were run to determine the effect on the spike of increasing the number of instantaneous observations at a given point. One case assumed 20 instantaneous observations, and the other assumed 200 instantaneous observations at each point. The spike disappeared in the latter case and was still present in the former. Since no discrepancies in the remaining determination are noted, the spike effect will not be discussed further.

It will be recalled that in the derivations which preceded this section, $\delta\phi_s$ was considered to be a variable. In reality the maximum value of $\delta\phi_s$ is approximately 12' of arc at 45° latitude and should remain relatively constant for a small change in latitude (that is, $\frac{\Delta(\delta\phi_s)}{\Delta\phi_s} \approx 0$ when $\Delta\phi_s$ is small).

To determine the effect of $\delta\phi_s$ on a determination, two cases were run in which $\delta\phi_s$ was first assumed to be a constant by equating its partials to zero and then a variable. Further, the approximations

$$\sin \delta\phi_s \approx \delta\phi_s$$

$$\cos \delta\phi_s \approx 1$$

were assumed valid in both cases. The results are shown in table III for the first determination of each case only, since it is representative of the remaining determinations.

TABLE III.- CALCULATED CHANGE IN DATUM FOR TWO CONDITIONS OF $\delta\phi_s$

Assumption	Δa , meters	Δf	$\Delta \xi_0$, radians	$\Delta \eta_0$, radians	$\Delta \delta N_s$, meters
$\delta\phi_s = \text{Constant}$ $\sin \delta\phi_s \approx \delta\phi_s$ $\cos \delta\phi_s \approx 1$	-71.126503	$-0.10408976 \times 10^{-4}$	3.4794727×10^{-5}	$-0.14433988 \times 10^{-4}$	4.3626973
$\delta\phi_s = \text{Variable}$ $\sin \delta\phi_s \approx \delta\phi_s$ $\cos \delta\phi_s \approx 1$	-221.53143	-.14628869	.51204351	-.12143458	-7.8855827

When the results of table III are compared with the desired results given in table I, it is seen that $\delta\phi_s$ has a large influence on the results, which was not expected, since $\delta\phi_s$ is so small and increases slowly. When $\delta\phi_s$ was considered to be constant the results were unacceptable although systematic. When approximations for the sine and cosine of $\delta\phi_s$ were substituted for their true values, the results were considerably better although not as accurate as hoped for. As a result, it is concluded that for best results, $\delta\phi_s$ cannot be assumed constant, and it is necessary to use the true sine and cosine of $\delta\phi_s$ in the methods of this paper.

Since it has been intended to show that the method is mathematically feasible and to show some of the conditions under which best results are obtained, results discussed thus far were computed in double precision on an electric data processing machine. Although it is not the purpose of this paper to do a statistical analysis of a practical application of the method, it was decided to generate some random numbers which could be added to the observed angle for the polar method. Thus, it could be determined if the method will converge or not with a large standard deviation of measurement.

The variance of the subtense angle is given by the relation

$$\sigma_T^2 = \sigma_I^2 + \sigma_H^2$$

where σ_I^2 is the variance of the instrument and the variance of the horizon uncertainty is given by the relation

$$\sigma_H^2 = \frac{H^2}{R_V^2 - R_M^2}$$

(from ref. 9). For the case of the polar angle, the relation for σ_H differs from that for the subtense angle in that σ_H^2 must be multiplied by the expression for $\partial\psi_{n,i}/\partial\alpha_{n,i}$ (see eq. (27)) to satisfy the condition that σ_H be a minimum for $\beta_i = \frac{\pi}{2}$ radians and for $\beta_i = \frac{3}{4}\pi$ radians. After a relation for the variance of the polar angle has been determined, random numbers can be generated for the purpose of testing the convergence of the method.

The case tested was identical to the one used throughout this paper with two exceptions. First, the incremental increase in R_{SV} was reduced to 2 kilometers to keep R_{SV} small. Secondly, the azimuth $Az_{g,n}$ and elevation γ_n were increased in steps of 10^{-6} radians. The incremental changes for each variable were applied simultaneously, and one observation was assumed for each point in space.

The test case was computed for two instrument variances. In both cases, the horizon uncertainty H was assumed to be 1.6 kilometers. For the first case, σ_I was taken to be 5×10^{-5} radians, and for the second case, σ_I was 5×10^{-4} radians. (In ref. 10, a horizon scanner is described with an accuracy of 0.1° of arc.)

The results in both cases were encouraging in that the sigmas for each unknown do, in fact, converge. However, convergence was slow. The results for both cases in which 30 000 observations were taken are shown in table IV. It should be noticed that the standard deviations for Δf , $\Delta \xi_0$, $\Delta \eta_0$, and $\Delta \delta N_0$ are still larger than the desired values (table I).

TABLE IV.- LEAST-SQUARES ESTIMATE AND STANDARD DEVIATION FOR 30 000 OBSERVATIONS

Standard deviation of instrument error, radians	Result	Δa , meters	Δf	$\Delta \xi_0$, radians	$\Delta \eta_0$, radians	$\Delta \delta N_0$, meters
5×10^{-5}	Estimate	-0286.20804	$-0.22345918 \times 10^{-4}$	$0.06674523 \times 10^{-3}$	$-0.05669926 \times 10^{-4}$	0190.94084
	σ	0069.21024	$0.09471739 \times 10^{-4}$	$0.11432682 \times 10^{-3}$	$0.16108228 \times 10^{-4}$	0124.95359
5×10^{-4}	Estimate	-0512.92349	$0.03380511 \times 10^{-4}$	$-0.26321512 \times 10^{-3}$	$0.08405173 \times 10^{-4}$	0069.04514
	σ	0388.95248	$0.47378258 \times 10^{-4}$	$0.43793034 \times 10^{-3}$	$0.51414847 \times 10^{-4}$	0348.57976

The results discussed here do not by any means constitute a complete error analysis of the method. No effort has been made to incorporate observations from more than one radar station. Further, it is not known how multiple horizon observations at each space point will affect the rate of convergence. The results do, however, demonstrate that the answers will converge for large variances in the observations when sufficient data have been collected.

CONCLUDING REMARKS

A theoretical method which can be used to derive an earth-centered geodetic datum has been derived and tested by mathematical simulation. Although a practical application of the method must necessarily include instrument errors and corrections for refraction and aberration, in this simulation such errors and corrections were omitted.

The results of the simulation for a test case proved it to be mathematically feasible. Further, when large variances in the observations were assumed, the method was found to converge, although slowly. Even so, additional statistical studies of the method are desirable. As for immediate usefulness, some of the required relations may have application in other areas of research -

such as, accuracy penalties resulting from assuming the earth to be spherical rather than oblate in navigation schemes and horizon - uncertainty studies.

Tests conducted to determine if the difference between the geodetic and geocentric latitudes $\delta\phi_s$ could be considered to be constant for small changes in latitude and sufficiently small to allow its sine and cosine to be approximately $\delta\phi_s$ and 1, respectively, gave unacceptable results. When the trigonometric functions were approximated only, the results were acceptable although the accuracy was not as good as hoped for. It is therefore concluded that for best results, no approximations can be made in $\delta\phi_s$, but acceptable results can be obtained by approximating the trigonometric functions of $\delta\phi_s$ only. Finally, under no conditions, could $\delta\phi_s$ be considered constant.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., December 10, 1965.

APPENDIX A

DERIVATION OF α

As explained in the text, if a satellite is assumed to be located at a point V_n in space and if by some suitable optical means (assuming no atmospheric refraction) a point T_i on the visible horizon is observed, the subtense angle $\alpha_{n,i}$ is given by the functional relation of equation (4). Furthermore, if the earth is assumed to be a smooth oblate ellipsoid, the line of sight $V_n T_i$ is tangent to the earth at T_i and perpendicular to the normal N_i . These two assumptions are the basis for the derivations which follow. In addition, the vehicle is considered to be frozen in space with respect to the earth, since $\alpha_{n,i}$ is independent of longitude as will be shown subsequently.

Consider figure A-1 in which X, Y, and Z are the axes of a right-handed, earth-centered, earth-fixed, rectangular Cartesian coordinate system

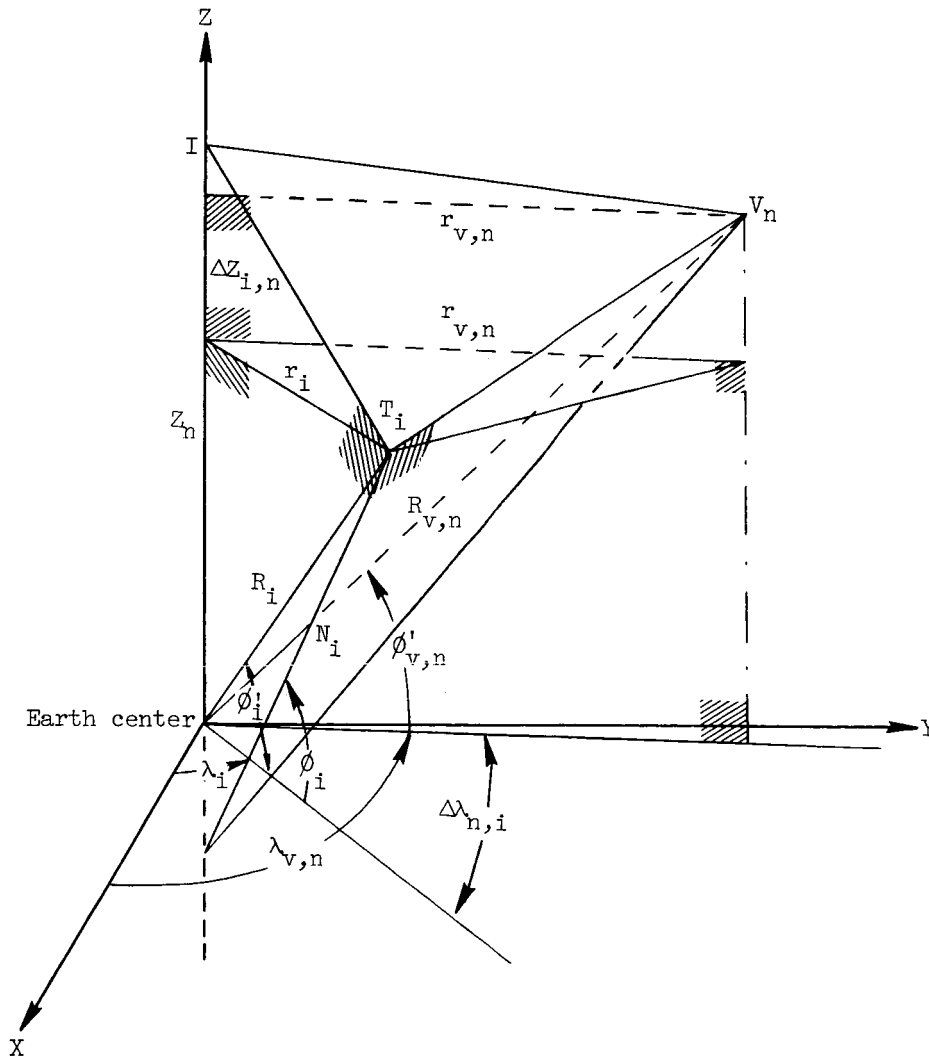


Figure A-1.- Geometrically determined longitude of a point on earth's visible horizon relative to location of a satellite.

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where the Z-axis is parallel with the earth's spin axis, the X-axis is parallel with the intersection of the Greenwich meridian and the equatorial planes, and the Y-axis completes a right-handed system; λ_i and $\lambda_{v,n}$ are the longitudes of T_i and V_n , respectively, and are conventionally measured from the Greenwich meridian positive in the eastward direction; ϕ_i is the geodetic latitude of T_i conventionally positive north from the equatorial plane; r_i and $r_{v,n}$ are radial distances from the earth's spin axis to T_i and V_n , respectively; the plane $T_i V_n I$ is tangent to the ellipsoid at T_i ; and N_i is the normal to the ellipsoid at T_i .

From figure A-1

$$N_i = \frac{r_i}{\cos \phi_i} \quad (A1)$$

$$\Delta\lambda_{n,i} = \lambda_{v,n} - \lambda_i$$

$$\Delta Z_{i,n} = Z_{v,n} - Z_i \quad (A2)$$

and

$$\overline{V_n T_i}^2 = r_{v,n}^2 + (N_i \sin \phi_i - Z_i + Z_{v,n})^2 - N_i^2 \quad (A3)$$

From the law of cosines,

$$\cos \Delta\lambda_{n,i} = \frac{\overline{V_n T_i}^2 - \Delta Z_{i,n}^2 - r_{v,n}^2 - r_i^2}{-2r_{v,n}r_i} \quad (A4)$$

Thus, after substituting equations (A1) and (A2) into (A3) and the resulting expression into (A4), the result reduces to

$$\cos \Delta\lambda_{n,i} = \frac{r_i - \Delta Z_{i,n} \left(\frac{\sin \phi_i}{\cos \phi_i} \right)}{r_{v,n}} \quad (A5)$$

But

$$\tan \phi'_i = (1 - e^2) \tan \phi_i$$

and, from figure A-1

$$\tan \phi'_i = \frac{z_i}{r_i}$$

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Therefore,

$$\frac{\sin \phi_i}{\cos \phi_i} = \frac{z_i}{r_i(1 - e^2)}$$

and, after substituting into equation (A5) and factoring,

$$\cos \Delta\lambda_{n,i} = \frac{r_i}{r_{v,n}} \left[1 - \frac{z_i \Delta z_{i,n}}{(1 - e^2) r_i^2} \right] \quad (A6)$$

From figure A-1,

$$r_i^2 = x_i^2 + y_i^2$$

But, from the equation of an oblate spheroid,

$$r_i^2 = x_i^2 + y_i^2 = \frac{a^2}{b^2} (b^2 - z_i^2) \quad (A7)$$

Therefore, when equations (A7) and (A2) are substituted in (A6) and the results reduced

$$\cos \Delta\lambda_{n,i} = \frac{a}{br_{v,n}} \frac{b^2 - z_{v,n} z_i}{(b^2 - z_i^2)^{1/2}} \quad (A8)$$

It is convenient here to define two additional coordinate systems, X', Y', Z' and X'', Y'', Z'' , which are, also, right-handed, rectangular, and cartesian. As shown in figure A-2, the X', Y', Z' system is earth-centered and the Z' -axis is parallel to the earth's spin axis and positive north, which is equivalent to the Z -axis of the X, Y, Z system. The Y' -axis lies in the equatorial plane and parallel with the intersection of the vehicle's meridian and the equatorial plane and is positive in the direction of the satellite. The X' -axis completes the right-handed system. The X'', Y'', Z'' coordinate system is vehicle-centered and defined in such a manner that the Y'' -axis is parallel to the radius vector $\bar{R}_{v,n}$ and positive in the same direction as $\bar{R}_{v,n}$; the Z'' -axis lies in the plane of observation normal to the Y'' -axis, positive in the positive direction of $\bar{D}_{n,i}$ and β_i degrees clockwise from the meridian of the vehicle; and the X'' -axis completes the right-hand system.

The coordinates of the satellite $x_{v,n}$, $y_{v,n}$, and $z_{v,n}$ in the X, Y, Z system become 0, $r_{v,n}$, and $z_{v,n}$ in the X', Y', Z' system where

$$r_{v,n} = R_{v,n} \cos \phi'_{v,n}$$

$$z_{v,n} = R_{v,n} \sin \phi'_{v,n}$$

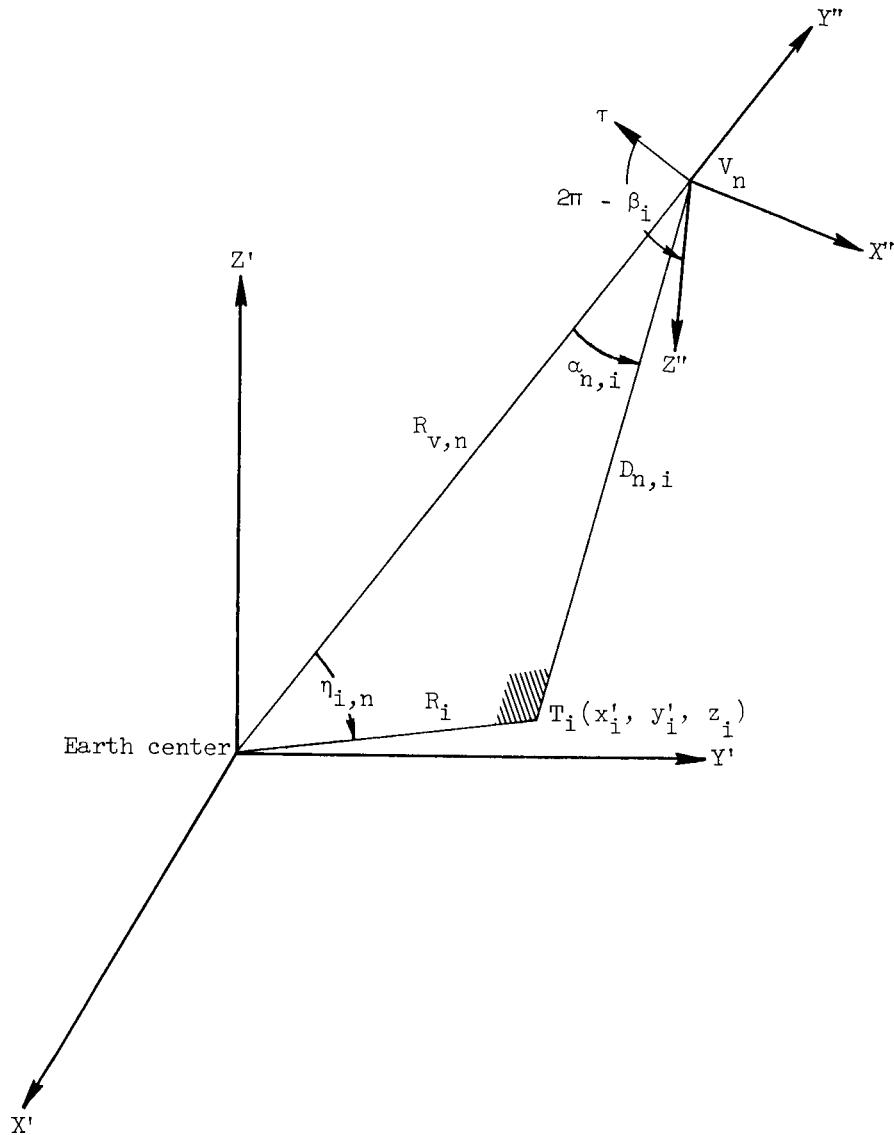


Figure A-2.- An optical observation relative to a satellite-centered coordinate system.

and the coordinates of the point of tangency T_i become x'_i , y'_i , and z'_i . Further, since the vector $\bar{D}_{n,i}$ ($\bar{D}_{n,i} = \bar{V}_n \bar{T}_i$ in fig. A-1) originates at the satellite and terminates at the point T_i on the visible horizon, the coordinates of T_i can be expressed in terms of $D_{n,i}$ and $\alpha_{n,i}$ as

$$x''_i = 0$$

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$$y_i'' = -D_{n,i} \cos \alpha_{n,i}$$

$$z_i'' = D_{n,i} \sin \alpha_{n,i}$$

Thus, from figure A-2

$$\bar{D}_{n,i} = -x_i' \bar{i} + (r_{v,n} - y_i') \bar{j} + (z_{v,n} - z_i') \bar{k}$$

or

$$\begin{aligned} D_{n,i}^2 &= \bar{D}_{n,i} \cdot \bar{D}_{n,i} = (x_i')^2 + (r_{v,n} - y_i')^2 + (z_{v,n} - z_i')^2 \\ &= R_i^2 + R_{v,n}^2 - 2(r_{v,n} y_i' + z_{v,n} z_i') \end{aligned} \quad (A9)$$

and, since

$$y_i' = r_i \cos \Delta\lambda_i = \frac{a^2}{b^2} \frac{1}{r_{v,n}} (b^2 - z_{v,n} z_i') \quad (A10)$$

equation (A9) becomes

$$D_{n,i}^2 = R_i^2 + R_{v,n}^2 - 2a^2 - 2z_{v,n} z_i' \left(1 - \frac{a^2}{b^2}\right) \quad (A11)$$

Furthermore, by the cosine law,

$$R_{v,n}^2 + D_{n,i}^2 - 2R_{v,n} D_{n,i} \cos \alpha_{n,i} = R_i^2$$

or

$$2R_{v,n} D_{n,i} \cos \alpha_{n,i} = R_{v,n}^2 + D_{n,i}^2 - R_i^2$$

which, upon substituting equation (A11) for $D_{n,i}^2$ and solving for $D_{n,i}$, reduces to

$$D_{n,i} = \frac{R_{v,n}^2 - a^2 - z_{v,n} z_i' \left(1 - \frac{a^2}{b^2}\right)}{R_{v,n} \cos \alpha_{n,i}} \quad (A12)$$

To find x_i' , y_i' , and z_i' , first rotate the X'', Y'', Z'' system about the Y'' -axis through the angle β_i , then rotate this system about the X'' -axis through the angle ϕ_v' , and, finally, translate the origin to the earth's center. In

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matrix notation

$$\begin{Bmatrix} x'_i \\ y'_i \\ z_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi'_{v,n} & -\sin \phi'_{v,n} \\ 0 & \sin \phi'_{v,n} & \cos \phi'_{v,n} \end{bmatrix} \begin{bmatrix} \cos \beta_i & 0 & -\sin \beta_i \\ 0 & 1 & 0 \\ \sin \beta_i & 0 & \cos \beta_i \end{bmatrix} \begin{Bmatrix} 0 \\ -D_{n,i} \cos \alpha_{n,i} \\ D_{n,i} \sin \alpha_{n,i} \end{Bmatrix} + \begin{Bmatrix} 0 \\ r_{v,n} \\ z_{v,n} \end{Bmatrix}$$

and

$$\left. \begin{aligned} x'_i &= -D_{n,i} \sin \alpha_{n,i} \sin \beta_i \\ y'_i &= -D_{n,i} (\cos \alpha_{n,i} \cos \phi'_{v,n} + \sin \alpha_{n,i} \cos \beta_i \sin \phi'_{v,n}) + r_{v,n} \\ z_i &= -D_{n,i} (\cos \alpha_{n,i} \sin \phi'_{v,n} - \sin \alpha_{n,i} \cos \beta_i \cos \phi'_{v,n}) + z_{v,n} \end{aligned} \right\} \quad (A13)$$

Substituting equation (A12) for $D_{n,i}$ in the expression for z_i and solving for z_i yields

$$z_i = \frac{z_{v,n} - \frac{R_{v,n}^2 - a^2}{R_{v,n}} (\sin \phi'_{v,n} - \tan \alpha_{n,i} \cos \beta_i \cos \phi'_{v,n})}{1 - \left(1 - \frac{a^2}{b^2}\right) (\sin^2 \phi'_{v,n} - \tan \alpha_{n,i} \cos \beta_i \sin \phi'_{v,n} \cos \phi'_{v,n})} \quad (A14)$$

which is the desired equation.

From figure A-2,

$$\bar{R}_{v,n} \times \bar{R}_i = \bar{\rho} R_{v,n} R_i \sin \eta_{i,n}$$

where $\bar{\rho}$ is a unit vector normal to the plane formed by $\bar{R}_{v,n}$ and \bar{R}_i . Thus,

$$1 = \bar{\rho} \cdot \bar{\rho} = \frac{(\bar{R}_{v,n} \times \bar{R}_i) \cdot (\bar{R}_{v,n} \times \bar{R}_i)}{(R_{v,n} R_i \sin \eta_{i,n})^2}$$

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which, after substituting the rectangular representations of $\bar{R}_{v,n}$ and \bar{R}_i , performing the indicated operations, and solving for $\sin^2 \eta_{i,n}$ becomes

$$\sin^2 \eta_{i,n} = \frac{r_{v,n}^2 z_i^2 - 2r_{v,n} z_{v,n} y_i' z_i + (y_i')^2 z_{v,n}^2 + z_{v,n}^2 (x_i')^2 + r_{v,n}^2 (x_i')^2}{R_i^2 R_{v,n}^2}$$

Now, by the sine law,

$$\sin^2 \alpha_{n,i} = \frac{R_i^2 \sin^2 \eta_{i,n}}{D_{n,i}^2}$$

Thus,

$$\sin^2 \alpha_{n,i} = \frac{r_{v,n}^2 z_i^2 - 2r_{v,n} z_{v,n} y_i' z_i + (y_i')^2 z_{v,n}^2 + z_{v,n}^2 (x_i')^2 + r_{v,n}^2 (x_i')^2}{D_{n,i}^2 R_{v,n}^2}$$

Substituting the identities

$$z_{v,n} = R_{v,n} \sin \phi_{v,n}'$$

$$r_{v,n} = R_{v,n} \cos \phi_{v,n}'$$

$$y_i' = \frac{a^2}{b^2} \frac{1}{r_{v,n}} (b^2 - z_{v,n} z_i)$$

$$x_i' = \sqrt{r_i^2 - (y_i')^2} = \left[\frac{a^2}{b^2} (b^2 - z_i^2) - (y_i')^2 \right]^{1/2}$$

$$\frac{a^2}{b^2} = \frac{1}{(1 - f)^2}$$

$$1 - \frac{a^2}{b^2} = - \frac{(2f - f^2)}{(1 - f)^2}$$

and rearranging the terms gives

$$\begin{aligned}
 \tan^2 \alpha_{n,i} \left(\frac{R_{v,n}^2 - a^2}{R_{v,n}} \right)^2 - \left(1 - \frac{a^2}{R_{v,n}^2} \right) a^2 \\
 = \left[\frac{a^2}{b^2} \sin^2 \phi'_{v,n} + \cos^2 \phi'_{v,n} - \tan^2 \alpha_{n,i} \sin^2 \phi'_{v,n} \left(1 - \frac{a^2}{b^2} \right) \right] \left(1 - \frac{a^2}{b^2} \right) z_i^2 \\
 + 2 \left[\left(\frac{R_{v,n}^2 - a^2}{R_{v,n}} \right) \tan^2 \alpha_{n,i} - \frac{a^2}{R_{v,n}} \right] \left(1 - \frac{a^2}{b^2} \right) z_i \sin \phi'_{v,n} \quad (A15)
 \end{aligned}$$

When equation (A14) is substituted for z_i in equation (A15) and the results are reduced and rearranged, an equation of the form

$$A_{n,i} \tan^2 \alpha_{n,i} + B_{n,i} \tan \alpha_{n,i} + C_{n,i} = 0$$

where

$$\begin{aligned}
 A_{n,i} &= \left(\frac{R_{v,n}^2 - a^2}{R_{v,n}} \right)^2 (1 - f)^4 + \left[R_{v,n}^2 (2f - f^2) \sin^2 \phi'_{v,n} \right. \\
 &\quad + 2(R_{v,n}^2 - a^2)(1 - f)^2 \left. \right] (2f - f^2) \sin^2 \phi'_{v,n} + \left[(2f - f^2) \sin^2 \phi'_{v,n} \right. \\
 &\quad + (1 - f)^2 \left(1 - \frac{a^2}{R_{v,n}^2} \right) \left. \right] (R_{v,n}^2 - a^2) (2f - f^2) \cos^2 \phi'_{v,n} \cos^2 \beta_i \quad (A16) \\
 B_{n,i} &= \left[(2f - f^2) \sin^2 \phi'_v + \left(1 - \frac{a^2}{R_{v,n}^2} \right) (1 - f)^2 \right] a^2 (2f - f^2) \sin 2\phi'_{v,n} \cos \beta_i \\
 C_{n,i} &= -a^2 \left[\left(1 - \frac{a^2}{R_{v,n}^2} \right) (1 - f)^4 + \left(2 - \frac{a^2}{R_{v,n}^2} \right) (1 - f)^2 (2f - f^2) \sin^2 \phi'_{v,n} \right. \\
 &\quad + (2f - f^2)^2 \sin^4 \phi'_{v,n} \left. \right]
 \end{aligned}$$

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is obtained. The desired expression for $\alpha_{n,i}$ is, therefore,

$$\alpha_{n,i} = \tan^{-1} \left(\frac{-B_{n,i} + \sqrt{B_{n,i}^2 - 4A_{n,i}C_{n,i}}}{2A_{n,i}} \right) \quad (A17)$$

The positive sign is chosen so that $\alpha_{n,i}$ is a minimum when $\beta_{n,i}$ is zero.

APPENDIX B

AN ERROR EQUATION FOR α

In the section entitled "Assumptions and General Theory," the general expression for the error equation of alpha (eq. (13)) was derived, and in appendix A, the expression for F was derived. To complete the solution of equation (13), it is necessary to derive first the expressions for the functions Φ , Λ , and P so that the proper partial derivatives for determining the coefficients of equation (11) can be obtained.

The partial derivatives of Φ , Λ , and P are unwieldy when no form of generalization is utilized. In an effort to generalize, the necessary partial differentiation of individual expressions will be indicated only in this section. The reader is, therefore, referred to appendix C for the necessary algebraic expressions which result from the partial differentiation.

Before Φ , Λ , and P can be derived, it is necessary to decide what method will be used to fix the position of the satellite. Because of the high accuracy obtainable, radar was chosen for this paper.

There are several techniques for fixing the instantaneous position of a space vehicle by radar observations in use today. Since it is impractical to discuss each technique and its individual application to the theory of this work, the conventional measurements of range $R_{sv,n}$, azimuth $Az_{g,n}$, and the elevation above the horizon γ_n are taken to be observables.

Of the three observables listed, only the range is independent of the deflection of the vertical. The dependence of $Az_{g,n}$ and γ_n on the components of the deflection of the vertical will be shown subsequently in this section.

The relations of equation (6) can be rewritten as follows:

$$\left. \begin{aligned} d\phi'_{v,n} &= \frac{\partial\Phi}{\partial a} da + \frac{\partial\Phi}{\partial f} df + \frac{\partial\Phi}{\partial\phi_s} d\phi_s + \frac{\partial\Phi}{\partial\lambda_s} d\lambda_s + \frac{\partial\Phi}{\partial h_s} dh_s + \frac{\partial\Phi}{\partial\gamma_n} d\gamma_n + \frac{\partial\Phi}{\partial Az_{sv,n}} dAz_{sv,n} \\ d\lambda_{v,n} &= \frac{\partial\Lambda}{\partial a} da + \frac{\partial\Lambda}{\partial f} df + \frac{\partial\Lambda}{\partial\phi_s} d\phi_s + \frac{\partial\Lambda}{\partial\lambda_s} d\lambda_s + \frac{\partial\Lambda}{\partial h_s} dh_s + \frac{\partial\Lambda}{\partial\gamma_n} d\gamma_n + \frac{\partial\Lambda}{\partial Az_{sv,n}} dAz_{sv,n} \\ dR_{v,n} &= \frac{\partial P}{\partial a} da + \frac{\partial P}{\partial f} df + \frac{\partial P}{\partial\phi_s} d\phi_s + \frac{\partial P}{\partial\lambda_s} d\lambda_s + \frac{\partial P}{\partial h_s} dh_s + \frac{\partial P}{\partial\gamma_n} d\gamma_n + \frac{\partial P}{\partial Az_{sv,n}} dAz_{sv,n} \end{aligned} \right\} \quad (B1)$$

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The partial derivatives of Φ , Λ , and P with respect to the range are zero, as explained previously in the text.

Thus, the rectangular coordinates of the radar station in the X, Y, Z coordinate system are

$$\left. \begin{aligned} x_S &= (N_S + h_S) \cos \phi_S \sin \lambda_S \\ y_S &= (N_S + h_S) \cos \phi_S \cos \lambda_S \\ z_S &= (N_S + h_S) \left[1 - \frac{N_S}{(N_S + h_S)} e^2 \right] \sin \phi_S \end{aligned} \right\} \quad (B2)$$

and

$$N_S = \frac{a}{W_S} \quad (B3)$$

is the expression for the magnitude of the normal to the ellipsoid where

$$W_S^2 = 1 - (2f - f^2) \sin^2 \phi_S$$

(ref. 1). Therefore,

$$\tan \phi'_S = \frac{z_S}{\sqrt{x_S^2 + y_S^2}} = \left[1 - \frac{N_S}{(N_S + h_S)} (2f - f^2) \right] \tan \phi_S \quad (B4)$$

where

$$2f - f^2 = e^2$$

is the relation between the geocentric and geodetic latitudes.

The total differential of equation (B3) is

$$dN_S = \frac{\partial N_S}{\partial a} da + \frac{\partial N_S}{\partial f} df + \frac{\partial N_S}{\partial \phi_S} d\phi_S \quad (B5)$$

and that of (B4) is

$$d\phi'_S = \frac{\partial \phi'_S}{\partial N_S} dN_S + \frac{\partial \phi'_S}{\partial h_S} dh_S + \frac{\partial \phi'_S}{\partial f} df + \frac{\partial \phi'_S}{\partial \phi_S} d\phi_S \quad (B6)$$

Thus, after substituting equation (B5) into (B6) and collecting coefficients of like terms

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$$d\phi'_s = \frac{\partial\phi'_s}{\partial N_s} \frac{\partial N_s}{\partial a} da + \left(\frac{\partial\phi'_s}{\partial N_s} \frac{\partial N_s}{\partial f} + \frac{\partial\phi'_s}{\partial f} \right) df + \left(\frac{\partial\phi'_s}{\partial\phi_s} + \frac{\partial\phi'_s}{\partial N_s} \frac{\partial N_s}{\partial\phi_s} \right) d\phi_s + \frac{\partial\phi'_s}{\partial h_s} dh_s \quad (B7)$$

In figure B-1, the relation at the radar station (or site) s of the normal to the ellipsoid \bar{N}_s , the geocentric radius vector of the site \bar{R}_s , and the range vector $\bar{R}_{sv,n}$ is shown. (The arrowheads indicate positive directions of the vectors.) The vectors \bar{N}_s and $\bar{R}_{sv,n}$ are in the plane of the paper so that the line segment sH represents the plane of the horizon and the vectors \bar{N}_s and \bar{R}_s are in the meridian plane.

If s is considered to be the center of a unit sphere, the spherical triangle DEF in figure B-1 can be formed where the angle at D is

$$D = \pi - Az_{g,n}$$

the angle at E is the spherical azimuth of the satellite $Az_{s,n}$, and $\delta\phi_s$ is given by the relation

$$\delta\phi_s = \phi_s - \phi'_s \quad (B8)$$

Thus, from the cosine law for spherical triangles,

$$\sin \gamma'_n = \cos \delta\phi_s \sin \gamma_n - \sin \delta\phi_s \cos \gamma_n \cos Az_{g,n} \quad (B9)$$

and from the law of sines

$$\sin Az_{s,n} = \frac{\cos \gamma_n}{\cos \gamma'_n} \sin Az_{g,n} \quad (B10)$$

The total differential of equation (B9) is

$$d\gamma'_n = \frac{\partial\gamma'_n}{\partial\gamma_n} d\gamma_n + \frac{\partial\gamma'_n}{\partial Az_{g,n}} dAz_{g,n} + \frac{\partial\gamma'_n}{\partial(\delta\phi_s)} d(\delta\phi_s) \quad (B11)$$

and that of (B10) is

$$dAz_{s,n} = \frac{\partial Az_{s,n}}{\partial Az_{g,n}} dAz_{g,n} + \frac{\partial Az_{s,n}}{\partial\gamma'_n} d\gamma'_n + \frac{\partial Az_{s,n}}{\partial\gamma_n} d\gamma_n \quad (B12)$$

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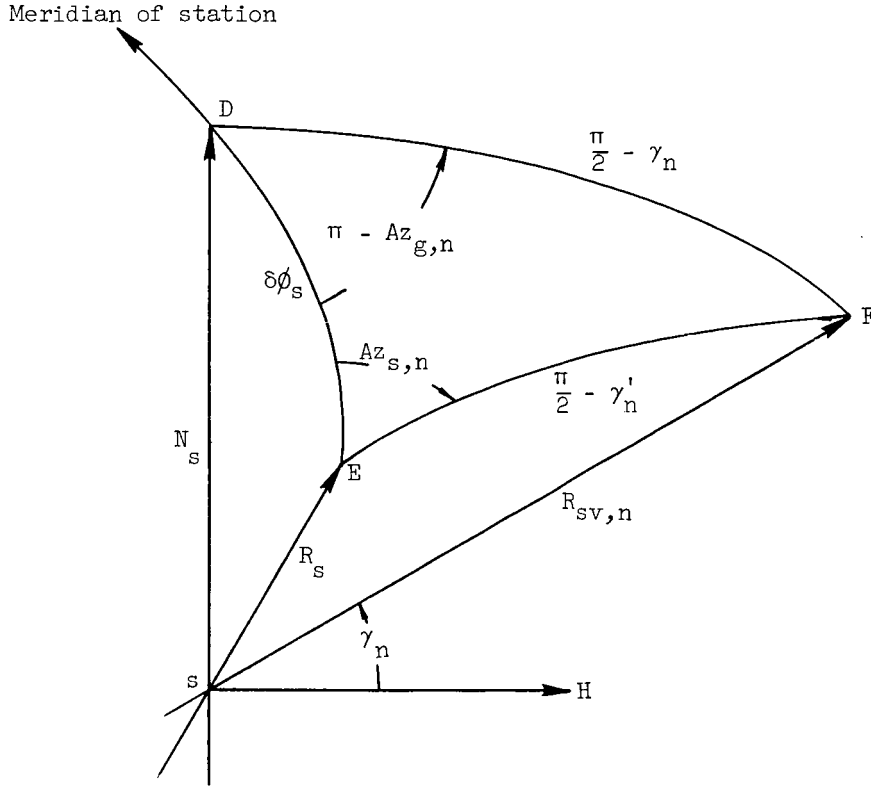


Figure B-1.- Relation of geodetic azimuth and elevation to equivalent spherical azimuth and elevation.

After substituting equation (B11) for $d\gamma_n'$ in (B12),

$$dAz_{s,n} = \left(\frac{\partial Az_{s,n}}{\partial Az_{g,n}} + \frac{\partial Az_{s,n}}{\partial \gamma_n'} \frac{\partial \gamma_n'}{\partial Az_{g,n}} \right) dAz_{g,n} + \left(\frac{\partial Az_{g,n}}{\partial \gamma_n} + \frac{\partial Az_{s,n}}{\partial \gamma_n'} \frac{\partial \gamma_n'}{\partial \gamma_n} \right) d\gamma_n + \frac{\partial Az_{s,n}}{\partial \gamma_n'} \frac{\partial \gamma_n'}{\partial (\delta\phi_s)} d(\delta\phi_s) \quad (B13)$$

As stated previously, $Az_{g,n}$ and γ_n are not independent of the deflection of the vertical. In reference 1 it is shown that

$$dAz_{g,n} = (\sin \phi_s - \cos Az_{g,n} \cos \phi_s \tan \gamma_n) d\lambda_s + \sin Az_{g,n} \tan \gamma_n d\phi_s \quad (B14)$$

The equivalent expression for γ_n must be derived.

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The normal to the best fitting ellipsoid will be deflected $\Delta\eta$ sec of arc in the meridian of the normal to the old ellipsoid and $\Delta\xi$ sec of arc in the prime vertical. It is now desired to determine the change in the elevation $d\gamma_n$ which will result.

Since the variables involved are arcs and will be measured at the common origin s (the radar station), figure B-2 represents the geometry of the problem to be solved. The angle included by the arcs $\Delta\eta$ and $\Delta\xi$ is a right angle. Thus, since $\Delta\eta$ and $\Delta\xi$ are small, the change in the deflection of the vertical ν can be written as

$$\nu^2 = \Delta\xi^2 + \Delta\eta^2$$

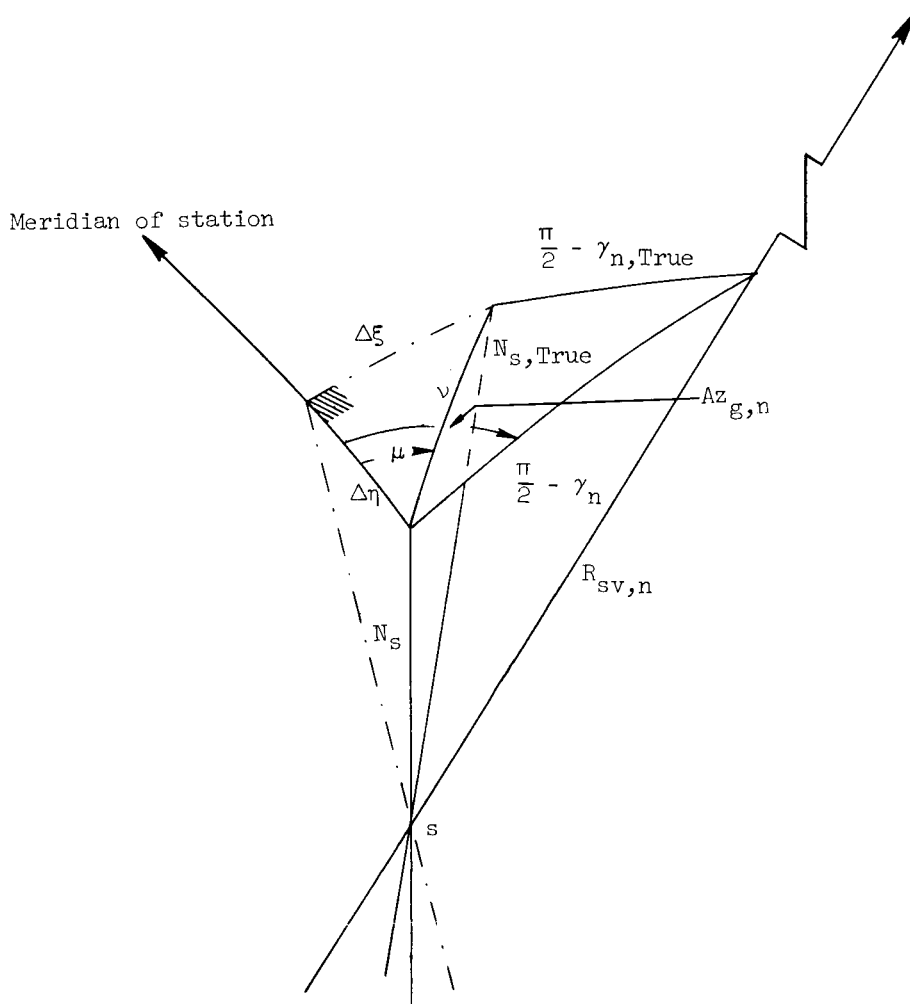


Figure B-2.- Effect on elevation resulting from a change in deflection of the vertical.

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Further, the law of cosines gives

$$\cos\left[\frac{\pi}{2} - (\gamma_n + d\gamma_n)\right] = \cos\left(\frac{\pi}{2} - \gamma_n\right)\cos v + \sin v \sin\left(\frac{\pi}{2} - \gamma_n\right)\cos(Az_{g,n} - \mu)$$

which reduces to

$$\sin(\gamma_n + d\gamma_n) = \sin \gamma_n \cos v + \sin v \cos \gamma_n (\cos Az_{g,n} \cos \mu + \sin Az_{g,n} \sin \mu)$$

where

$$\mu = \tan^{-1} \frac{\Delta\xi}{\Delta\eta} = \sin^{-1} \frac{\Delta\xi}{v} = \cos^{-1} \frac{\Delta\eta}{v}$$

In reference 1, $\Delta\xi$ and $\Delta\eta$ are related to the corresponding changes in latitude and longitude by the expressions

$$\Delta\xi = d\lambda_s \cos \phi_s$$

$$\Delta\eta = d\phi_s$$

(In ref. 1, $\Delta\xi$ and $\Delta\eta$ are defined to be positive in directions opposite to those used here. Thus, the signs in the above expressions must differ from those in the reference.) Thus, since v is small and $d\gamma_n$ will be small,

$$d\gamma_n = \cos Az_{g,n} d\phi_s + \sin Az_{g,n} \cos \phi_s d\lambda_s \quad (B15)$$

For expediency, the coefficients of $d\phi_s$ and $d\lambda_s$ in equations (B14) and (B15) are defined as

$$\left. \begin{aligned} \sin Az_{g,n} \tan \gamma_n &= \frac{\partial Az_{g,n}}{\partial \phi_s} \\ \cos Az_{g,n} &= \frac{\partial \gamma_n}{\partial \phi_s} \end{aligned} \right\} \quad (B16)$$

$$\sin \phi_s - \cos Az_{g,n} \cos \phi_s \tan \gamma_n = \frac{\partial Az_{g,n}}{\partial \lambda_s}$$

$$\sin Az_{g,n} \cos \phi_s = \frac{\partial \gamma_n}{\partial \lambda_s}$$

Thus,

$$dAz_{g,n} = \frac{\partial Az_{g,n}}{\partial \phi_s} d\phi_s + \frac{\partial Az_{g,n}}{\partial \lambda_s} d\lambda_s \quad (B17)$$

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and

$$d\gamma_n = \frac{\partial \gamma_n}{\partial \phi_s} d\phi_s + \frac{\partial \gamma_n}{\partial \lambda_s} d\lambda_s \quad (B18)$$

By definition

$$\delta\phi_s = \phi_s - \phi'_s$$

Thus,

$$d(\delta\phi_s) = \frac{\partial(\delta\phi_s)}{\partial \phi_s} d\phi_s + \frac{\partial(\delta\phi_s)}{\partial \phi'_s} d\phi'_s = d\phi_s - d\phi'_s$$

and, after substituting equation (B7) for $d\phi'_s$

$$d(\delta\phi_s) = -\frac{\partial \phi'_s}{\partial a} da - \frac{\partial \phi'_s}{\partial f} df - \frac{\partial \phi'_s}{\partial h_s} dh_s + \left(1 - \frac{\partial \phi'_s}{\partial \phi_s}\right) d\phi_s \quad (B19)$$

Finally, substituting equations (B19), (B18), and (B17) into (B11) and (B13) and collecting coefficients of like terms give

$$d\gamma'_n = \frac{\partial \gamma'_n}{\partial a} da + \frac{\partial \gamma'_n}{\partial f} df + \frac{\partial \gamma'_n}{\partial \phi_s} d\phi_s + \frac{\partial \gamma'_n}{\partial \lambda_s} d\lambda_s + \frac{\partial \gamma'_n}{\partial h_s} dh_s \quad (B20)$$

where

$$\left. \begin{aligned} \frac{\partial \gamma'_n}{\partial a} &= -\frac{\partial \gamma'_n}{\partial(\delta\phi_s)} \frac{\partial \phi'_s}{\partial a} \\ \frac{\partial \gamma'_n}{\partial f} &= -\frac{\partial \gamma'_n}{\partial(\delta\phi_s)} \frac{\partial \phi'_s}{\partial f} \\ \frac{\partial \gamma'_n}{\partial h_s} &= -\frac{\partial \gamma'_n}{\partial(\delta\phi_s)} \frac{\partial \phi'_s}{\partial h_s} \\ \frac{\partial \gamma'_n}{\partial \phi_s} &= \frac{\partial \gamma'_n}{\partial \gamma_n} \frac{\partial \gamma_n}{\partial \phi_s} + \frac{\partial \gamma'_n}{\partial A_{z_{g,n}}} \frac{\partial A_{z_{g,n}}}{\partial \phi_s} + \frac{\partial \gamma'_n}{\partial(\delta\phi_s)} \left(1 - \frac{\partial \phi'_s}{\partial \phi_s}\right) \\ \frac{\partial \gamma'_n}{\partial \lambda_s} &= \frac{\partial \gamma'_n}{\partial \gamma_n} \frac{\partial \gamma_n}{\partial \lambda_s} + \frac{\partial \gamma'_n}{\partial A_{z_{g,n}}} \frac{\partial A_{z_{g,n}}}{\partial \lambda_s} \end{aligned} \right\} \quad (B21)$$

and

$$dA_{z_{s,n}} = \frac{\partial A_{z_{s,n}}}{\partial a} da + \frac{\partial A_{z_{s,n}}}{\partial f} df + \frac{\partial A_{z_{s,n}}}{\partial \phi_s} d\phi_s + \frac{\partial A_{z_{s,n}}}{\partial \lambda_s} d\lambda_s + \frac{\partial A_{z_{s,n}}}{\partial h_s} dh_s \quad (B22)$$

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where

$$\left. \begin{aligned}
 \frac{\partial Az_{s,n}}{\partial a} &= - \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial (\delta \phi_s)} \frac{\partial \phi'_s}{\partial a} \\
 \frac{\partial Az_{s,n}}{\partial f} &= - \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial (\delta \phi_s)} \frac{\partial \phi'_s}{\partial f} \\
 \frac{\partial Az_{s,n}}{\partial h_s} &= - \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial (\delta \phi_s)} \frac{\partial \phi'_s}{\partial h_s} \\
 \frac{\partial Az_{s,n}}{\partial \phi_s} &= \left(\frac{\partial Az_{g,n}}{\partial \gamma_n} + \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial \gamma_n} \right) \frac{\partial \gamma_n}{\partial \phi_s} + \left(\frac{\partial Az_{s,n}}{\partial Az_{g,n}} + \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial Az_{g,n}} \right) \frac{\partial Az_{g,n}}{\partial \phi_s} \\
 &\quad + \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial (\delta \phi_s)} \left(1 - \frac{\partial \phi'_s}{\partial \phi_s} \right) \\
 \frac{\partial Az_{s,n}}{\partial \lambda_s} &= \left(\frac{\partial Az_{g,n}}{\partial \gamma_n} + \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial \gamma_n} \right) \frac{\partial \gamma_n}{\partial \lambda_s} + \left(\frac{\partial Az_{s,n}}{\partial Az_{g,n}} + \frac{\partial Az_{s,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial Az_{g,n}} \right) \frac{\partial Az_{g,n}}{\partial \lambda_s}
 \end{aligned} \right\} \quad (B23)$$

The rectangular coordinates of the radar station are given in terms of the geodetic latitude, longitude, and station height by equation (B2). The rectangular coordinates are given as functions of spherical coordinates by

$$\left. \begin{aligned}
 x_s &= R_s \cos \phi'_s \sin \lambda_s \\
 y_s &= R_s \cos \phi'_s \cos \lambda_s \\
 z_s &= R_s \sin \phi'_s
 \end{aligned} \right\} \quad (B24)$$

If the x_s coordinates of equations (B2) and (B24) are equated, the resulting expression solved for R_s would be

$$R_s = (N_s + h_s) \frac{\cos \phi_s}{\cos \phi'_s} \quad (B25)$$

The same expression would result with the y_s coordinate. The z_s coordinate would, however, give a different expression for R_s , but the numerical value would be the same.

The total differential of R_s is

$$dR_s = \frac{\partial R_s}{\partial N_s} dN_s + \frac{\partial R_s}{\partial h_s} dh_s + \frac{\partial R_s}{\partial \phi_s} d\phi_s + \frac{\partial R_s}{\partial \phi'_s} d\phi'_s \quad (B26)$$

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Since dN_s and $d\phi'_s$ are given by equations (B5) and (B6), respectively, after substituting into equation (B26) and collecting the coefficients of like terms,

$$dR_s = \left(\frac{\partial R_s}{\partial N_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \right) \frac{\partial N_s}{\partial a} da + \left(\frac{\partial R_s}{\partial h_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial h_s} \right) dh_s + \left[\left(\frac{\partial R_s}{\partial N_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \right) \frac{\partial N_s}{\partial \phi'_s} + \frac{\partial R_s}{\partial \phi'_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial \phi'_s} \right] d\phi'_s + \left[\left(\frac{\partial R_s}{\partial N_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \right) \frac{\partial N_s}{\partial f} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial f} \right] df \quad (B27)$$

Consider figure B-3 in which the polar triangle $(NP)(V'_{n,s})(s)$ is formed by the intersection of the meridian planes of the radar station and vehicle and the plane containing the geocentric radii of the vehicle and the radar station with a sphere of radius R_s . The angle at NP is

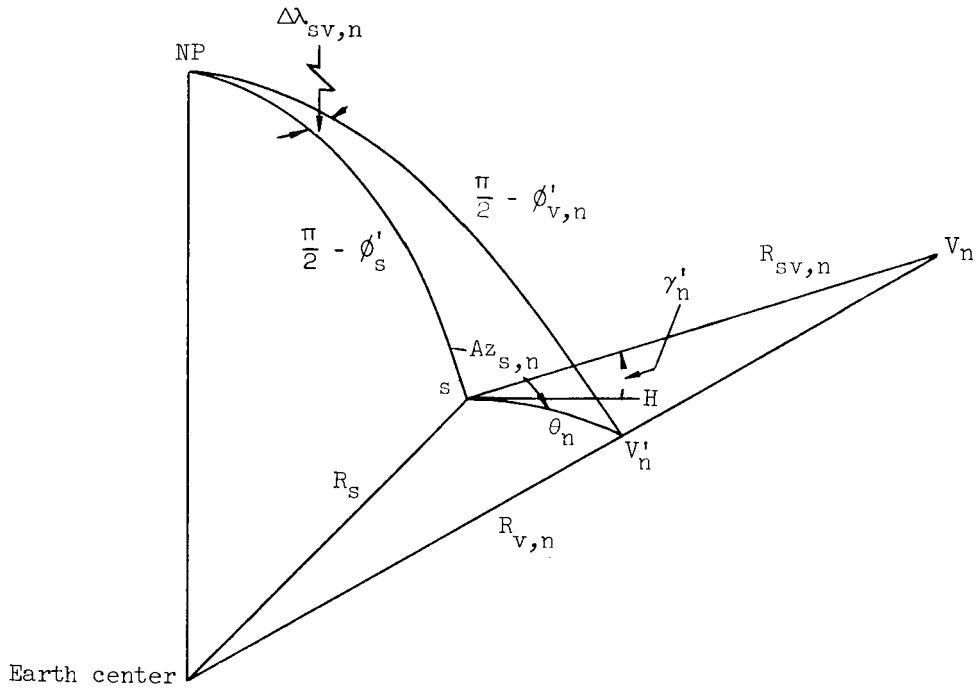


Figure B-3.- Spherical coordinates of a satellite relative to a radar station.

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$$\Delta\lambda_{sv,n} = \lambda_{v,n} - \lambda_s$$

and that at s is the spherical azimuth $Az_{s,n}$ of the satellite. Further, the arcs \widehat{NPs} and \widehat{NPV}'_n are

$$\widehat{NPs} = \frac{\pi}{2} - \phi'_s$$

and

$$\widehat{NPV}'_n = \frac{\pi}{2} - \phi'_{v,n}$$

The arc \widehat{sV}'_n is defined as θ_n and, from the plane triangle of figure B-3, it is seen that

$$\tan \theta_n = \frac{R_{sv,n} \cos \gamma'_n}{R_s + R_{sv,n} \sin \gamma'_n} \quad (B28)$$

Since the radar observation of range is considered herein to be errorless, the total differential of θ_n is

$$d\theta_n = \frac{\partial \theta_n}{\partial R_s} dR_s + \frac{\partial \theta_n}{\partial \gamma'_n} d\gamma'_n \quad (B29)$$

But, $d\gamma'_n$ is given by equation (B20) and dR_s is given by (B27). Thus, after substituting and collecting terms

$$d\theta_n = \frac{\partial \theta_n}{\partial a} da + \frac{\partial \theta_n}{\partial f} df + \frac{\partial \theta_n}{\partial \phi_s} d\phi_s + \frac{\partial \theta_n}{\partial \lambda_s} d\lambda_s + \frac{\partial \theta_n}{\partial h_s} dh_s \quad (B30)$$

where

$$\left. \begin{aligned} \frac{\partial \theta_n}{\partial a} &= \frac{\partial \theta_n}{\partial R_s} \left(\frac{\partial R_s}{\partial N_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \right) \frac{\partial N_s}{\partial a} + \frac{\partial \gamma'_n}{\partial a} \frac{\partial \theta_n}{\partial \gamma'_n} \\ \frac{\partial \theta_n}{\partial f} &= \left[\left(\frac{\partial R_s}{\partial N_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \right) \frac{\partial N_s}{\partial f} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial f} \right] \frac{\partial \theta_n}{\partial R_s} + \frac{\partial \gamma'_n}{\partial f} \frac{\partial \theta_n}{\partial \gamma'_n} \\ \frac{\partial \theta_n}{\partial \phi_s} &= \left[\left(\frac{\partial R_s}{\partial N_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial N_s} \right) \frac{\partial N_s}{\partial \phi_s} + \frac{\partial R_s}{\partial \phi_s} + \frac{\partial R_s}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial \phi_s} \right] \frac{\partial \theta_n}{\partial R_s} + \frac{\partial \gamma'_n}{\partial \phi_s} \frac{\partial \theta_n}{\partial \gamma'_n} \end{aligned} \right\} \quad (B31)$$

(Equation continued on next page)

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$$\left. \begin{aligned} \frac{\partial \theta_n}{\partial \lambda_s} &= \frac{\partial \gamma_n'}{\partial \lambda_s} \frac{\partial \theta_n}{\partial \gamma_n'} \\ \frac{\partial \theta_n}{\partial h_s} &= \frac{\partial \theta_n}{\partial R_s} \left(\frac{\partial R_s}{\partial h_s} + \frac{\partial R_s}{\partial \phi_s'} \frac{\partial \phi_s'}{\partial h_s} \right) + \frac{\partial \gamma_n'}{\partial h_s} \frac{\partial \theta_n}{\partial \gamma_n'} \end{aligned} \right\} \quad (B31)$$

Two additional expressions which can be obtained from the spherical triangle of figure B-3 are

$$\sin \phi_{v,n}' = \cos \theta_n \sin \phi_s' + \sin \theta_n \cos \phi_s' \cos Az_{s,n} \quad (B32)$$

and

$$\sin \Delta \lambda_{sv,n} = \frac{\sin Az_{s,n}}{\cos \phi_{v,n}'} \sin \theta_n \quad (B33)$$

The total differential of equation (B32) is

$$d\phi_{v,n}' = \frac{\partial \phi_{v,n}'}{\partial \phi_s'} d\phi_s' + \frac{\partial \phi_{v,n}'}{\partial \theta_n} d\theta_n + \frac{\partial \phi_{v,n}'}{\partial Az_{s,n}} dAz_{s,n} \quad (B34)$$

and that of (B33) is

$$d\Delta \lambda_{sv,n} = \frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} d\theta_n + \frac{\partial \Delta \lambda_{sv,n}}{\partial \phi_{v,n}'} d\phi_{v,n}' + \frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} dAz_{s,n} \quad (B35)$$

The differentials $d\phi_s'$, $d\theta_n$, and $dAz_{s,n}$ are given, respectively, by equations (B7), (B30), and (B22). Therefore, after substituting into equation (B34) and collecting coefficients of like terms,

$$d\phi_{v,n}' = \phi_{a,n}' da + \phi_{f,n}' df + \phi_{\phi_s,n}' d\phi_s + \phi_{\lambda_s,n}' d\lambda_s + \phi_{h_s,n}' dh_s \quad (B36)$$

where

$$\left. \begin{aligned} \phi_{a,n}' &= \frac{\partial \phi_{v,n}'}{\partial \phi_s'} \frac{\partial \phi_s'}{\partial N_s} \frac{\partial N_s}{\partial a} + \frac{\partial \phi_{v,n}'}{\partial \theta_n} \frac{\partial \theta_n}{\partial a} + \frac{\partial \phi_{v,n}'}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial a} \\ \phi_{f,n}' &= \frac{\partial \phi_{v,n}'}{\partial \phi_s'} \left(\frac{\partial \phi_s'}{\partial N_s} \frac{\partial N_s}{\partial f} + \frac{\partial \phi_s'}{\partial f} \right) + \frac{\partial \phi_{v,n}'}{\partial \theta_n} \frac{\partial \theta_n}{\partial f} + \frac{\partial \phi_{v,n}'}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial f} \\ \phi_{\phi_s,n}' &= \frac{\partial \phi_{v,n}'}{\partial \phi_s'} \left(\frac{\partial \phi_s'}{\partial N_s} \frac{\partial N_s}{\partial \phi_s} + \frac{\partial \phi_s'}{\partial \phi_s} \right) + \frac{\partial \phi_{v,n}'}{\partial \theta_n} \frac{\partial \theta_n}{\partial \phi_s} + \frac{\partial \phi_{v,n}'}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \phi_s} \end{aligned} \right\} \quad (B37)$$

(Equation continued on next page)

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$$\left. \begin{aligned} \Phi'_{\lambda_s, n} &= \frac{\partial \phi'_{v, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \lambda_s} + \frac{\partial \phi'_{v, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial \lambda_s} \\ \Phi'_{h_s, n} &= \frac{\partial \phi'_{v, n}}{\partial \phi'_s} \frac{\partial \phi'_s}{\partial h_s} + \frac{\partial \phi'_{v, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial h_s} + \frac{\partial \phi'_{v, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial h_s} \end{aligned} \right\} \quad (B37)$$

Before the functional coefficients of the $\lambda_{v, n}$ error equation can be derived, it is necessary to refer to the definition of $\Delta\lambda_{sv, n}$. By definition

$$\Delta\lambda_{sv, n} = \lambda_{v, n} - \lambda_s$$

Thus,

$$d\Delta\lambda_{sv, n} = d\lambda_{v, n} - d\lambda_s \quad (B38)$$

and

$$d\lambda_{v, n} = d\Delta\lambda_{sv, n} + d\lambda_s \quad (B39)$$

After substituting equation (B35) into (B39), by a procedure similar to that for deriving the coefficients of $d\phi'_{v, n}$,

$$d\lambda_{v, n} = \Lambda'_{a, n} da + \Lambda'_{f, n} df + \Lambda'_{\phi_s, n} d\phi_s + \Lambda'_{\lambda_s, n} d\lambda_s + \Lambda'_{h_s, n} dh_s \quad (B40)$$

where

$$\left. \begin{aligned} \Lambda'_{a, n} &= \Phi'_{a, n} \frac{\partial \Delta\lambda_{sv, n}}{\partial \phi'_{v, n}} + \frac{\partial \Delta\lambda_{sv, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial a} + \frac{\partial \Delta\lambda_{sv, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial a} \\ \Lambda'_{f, n} &= \Phi'_{f, n} \frac{\partial \Delta\lambda_{sv, n}}{\partial \phi'_{v, n}} + \frac{\partial \Delta\lambda_{sv, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial f} + \frac{\partial \Delta\lambda_{sv, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial f} \\ \Lambda'_{\phi_s, n} &= \Phi'_{\phi_s, n} \frac{\partial \Delta\lambda_{sv, n}}{\partial \phi'_{v, n}} + \frac{\partial \Delta\lambda_{sv, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \phi_s} + \frac{\partial \Delta\lambda_{sv, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial \phi_s} \\ \Lambda'_{\lambda_s, n} &= 1 + \Phi'_{\lambda_s, n} \frac{\partial \Delta\lambda_{sv, n}}{\partial \phi'_{v, n}} + \frac{\partial \Delta\lambda_{sv, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial \lambda_s} + \frac{\partial \Delta\lambda_{sv, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial \lambda_s} \\ \Lambda'_{h_s, n} &= \Phi'_{h_s, n} \frac{\partial \Delta\lambda_{sv, n}}{\partial \phi'_{v, n}} + \frac{\partial \Delta\lambda_{sv, n}}{\partial \theta_n} \frac{\partial \theta_n}{\partial h_s} + \frac{\partial \Delta\lambda_{sv, n}}{\partial Az_{s, n}} \frac{\partial Az_{s, n}}{\partial h_s} \end{aligned} \right\} \quad (B41)$$

To complete the derivation of the coefficients of equation (8), an expression is needed for $dR_{v, n}$. From the plane triangle of figure B-3

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$$R_{v,n} = R_{sv,n} \frac{\cos \gamma'_n}{\sin \theta_n} \quad (B42)$$

But

$$\sin \theta_n = \frac{\cos \phi'_{v,n} \sin \Delta\lambda_{sv,n}}{\sin Az_{s,n}}$$

Thus,

$$R_{v,n} = R_{sv,n} \frac{\cos \gamma'_n \sin Az_{s,n}}{\cos \phi'_{v,n} \sin \Delta\lambda_{sv,n}} \quad (B43)$$

The differential of equation (B43) is

$$dR_{v,n} = \frac{\partial R_{v,n}}{\partial \gamma'_n} d\gamma'_n + \frac{\partial R_{v,n}}{\partial Az_{s,n}} dAz_{s,n} + \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} d\phi'_{v,n} + \frac{\partial R_{v,n}}{\partial \Delta\lambda_{sv,n}} d\Delta\lambda_{sv,n} \quad (B44)$$

After substituting for $d\Delta\lambda_{sv,n}$, $dAz_{s,n}$, $d\phi'_{v,n}$, and $d\gamma'_n$,

$$dR_{v,n} = P'_{a,n} da + P'_{f,n} df + P'_{\phi_s,n} d\phi_s + P'_{\lambda_s,n} d\lambda_s + P'_{h_s,n} dh_s \quad (B45)$$

where

$$\left. \begin{aligned} P'_{a,n} &= \Phi'_{a,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{a,n} \frac{\partial R_{v,n}}{\partial \Delta\lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial a} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial a} \\ P'_{f,n} &= \Phi'_{f,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{f,n} \frac{\partial R_{v,n}}{\partial \Delta\lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial f} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial f} \\ P'_{\phi_s,n} &= \Phi'_{\phi_s,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{\phi_s,n} \frac{\partial R_{v,n}}{\partial \Delta\lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial \phi_s} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \phi_s} \\ P'_{\lambda_s,n} &= \Phi'_{\lambda_s,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + (\Lambda'_{\lambda_s,n} - 1) \frac{\partial R_{v,n}}{\partial \Delta\lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial \lambda_s} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial \lambda_s} \\ P'_{h_s,n} &= \Phi'_{h_s,n} \frac{\partial R_{v,n}}{\partial \phi'_{v,n}} + \Lambda'_{h_s,n} \frac{\partial R_{v,n}}{\partial \Delta\lambda_{sv,n}} + \frac{\partial R_{v,n}}{\partial \gamma'_n} \frac{\partial \gamma'_n}{\partial h_s} + \frac{\partial R_{v,n}}{\partial Az_{s,n}} \frac{\partial Az_{s,n}}{\partial h_s} \end{aligned} \right\} \quad (B46)$$

Now that the coefficients for equations (8) have been determined, it remains only to determine the coefficients of (12). It will be recalled that

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$$\alpha_{n,i} = F = \tan^{-1} \left(\frac{-B_{n,i} + \sqrt{B_{n,i}^2 - 4A_{n,i}C_{n,i}}}{2A_{n,i}} \right)$$

where $A_{n,i}$, $B_{n,i}$, and $C_{n,i}$ are given by equation (A16). As a result,

$$\left. \begin{aligned} \left(\frac{\partial F}{\partial a} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial a} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial a} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial a} \\ \left(\frac{\partial F}{\partial f} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial f} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial f} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial f} \\ \left(\frac{\partial F}{\partial \phi'_{v,n}} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial \phi'_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial \phi'_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial \phi'_{v,n}} \\ \left(\frac{\partial F}{\partial R_{v,n}} \right)_{n,i} &= \frac{\partial \alpha_{n,i}}{\partial A_{n,i}} \frac{\partial A_{n,i}}{\partial R_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial B_{n,i}} \frac{\partial B_{n,i}}{\partial R_{v,n}} + \frac{\partial \alpha_{n,i}}{\partial C_{n,i}} \frac{\partial C_{n,i}}{\partial R_{v,n}} \end{aligned} \right\} \quad (B47)$$

which are the desired expressions.

The error equation for $\alpha_{n,i}$ is given in a general form by equation (13). As stated previously, the coefficients of equation (13) are functions of those of (8), (9), and (12).

In this section the coefficients of equations (8) and (12) were derived as functions of indicated partial derivatives and are given in equations (B37), (B41), (B46), and (B47). The indicated operations have been executed and the results are given in appendix C. To obtain the algebraic expressions for equations (B37), (B41), (B46), and (B47), it is required only to make necessary substitutions.

APPENDIX C

ANALYTICAL EXPRESSIONS FOR VARIOUS PARTIAL DERIVATIVES

In the section entitled "The Error Equation for α " the coefficients of equations (8) and (13) were derived as functions of partial derivatives. In this appendix the partial derivatives of those expressions on which the coefficients are dependent will be given.

The partial derivatives of expressions (B3), (B4), (B9), (B10), (B25), (B28), (B32), (B33), and (B43) are as follows:

$$\frac{\partial N_s}{\partial \phi_s} = (2f - f^2) \frac{N_s}{W_s^2} \sin \phi_s \cos \phi_s$$

$$\frac{\partial N_s}{\partial a} = \frac{N_s}{a}$$

$$\frac{\partial N_s}{\partial f} = (1 - f) \frac{N_s}{W_s^2} \sin^2 \phi_s$$

$$\frac{\partial \phi'_s}{\partial h_s} = (2f - f^2) \frac{N_s}{(N_s + h_s)^2} \tan \phi_s \cos^2 \phi'_s$$

$$\frac{\partial \phi'_s}{\partial N_s} = -(2f - f^2) \frac{h_s}{(N_s + h_s)^2} \tan \phi_s \cos^2 \phi'_s$$

$$\frac{\partial \phi'_s}{\partial \phi_s} = \left[1 - \frac{N_s}{N_s + h_s} (2f - f^2) \right] \frac{\cos^2 \phi'_s}{\cos^2 \phi_s}$$

$$\frac{\partial \phi'_s}{\partial f} = -2(1 - f) \frac{N_s}{N_s + h_s} \tan \phi_s \cos^2 \phi'_s$$

$$\frac{\partial R_s}{\partial h_s} = \frac{R_s}{N_s + h_s}$$

$$\frac{\partial R_s}{\partial N_s} = \frac{R_s}{N_s + h_s}$$

$$\frac{\partial R_s}{\partial \phi_s} = R_s \tan \phi'_s$$

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$$\frac{\partial R_s}{\partial \phi_s} = -R_s \tan \phi_s$$

$$\frac{\partial \theta_n}{\partial R_s} = \frac{-\sin^2 \theta_n}{R_{sv,n} \cos \gamma'_n}$$

$$\frac{\partial \theta_n}{\partial \gamma'_n} = -\left(\frac{R_s}{R_{sv,n}} \sin \gamma_n + 1\right) \frac{\sin^2 \theta_n}{\cos^2 \gamma'_n}$$

$$\frac{\partial \gamma'_n}{\partial \delta \phi_s} = - \frac{\sin \delta \phi_s \sin \gamma_n + \cos \delta \phi_s \cos \gamma_n \cos Az_{g,n}}{\cos \gamma'_n}$$

$$\frac{\partial \gamma'_n}{\partial \gamma_n} = \frac{\cos \delta \phi_s \cos \gamma_n + \sin \delta \phi_s \sin \gamma_n \cos Az_{g,n}}{\cos \gamma'_n}$$

$$\frac{\partial \gamma'_n}{\partial Az_{g,n}} = \frac{\sin \delta \phi_s \cos \gamma_n \sin Az_{g,n}}{\cos \gamma'_n}$$

$$\frac{\partial Az_{s,n}}{\partial \gamma'_n} = \tan Az_{s,n} \tan \gamma'_n$$

$$\frac{\partial Az_{s,n}}{\partial \gamma_n} = -\tan Az_{s,n} \tan \gamma_n$$

$$\frac{\partial Az_{s,n}}{\partial Az_{g,n}} = \tan Az_{s,n} \cot Az_{g,n}$$

$$\frac{\partial \phi'_{v,n}}{\partial \phi'_s} = \frac{\cos \theta_n \cos \phi'_s - \sin \theta_n \sin \phi'_s \cos Az_{s,n}}{\cos \phi'_{v,n}}$$

$$\frac{\partial \phi'_{v,n}}{\partial \theta_n} = \frac{-\sin \phi'_s \sin \theta_n + \cos \phi'_s \cos \theta_n \cos Az_{s,n}}{\cos \phi'_{v,n}}$$

$$\frac{\partial \phi'_{v,n}}{\partial Az_{s,n}} = \frac{-\sin \theta_n \cos \phi'_s \sin Az_{s,n}}{\cos \phi'_{v,n}}$$

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$$\frac{\partial \Delta \lambda_{sv,n}}{\partial \theta_n} = \tan \Delta \lambda_{sv,n} \cot \theta_n$$

$$\frac{\partial \Delta \lambda_{sv,n}}{\partial Az_{s,n}} = \tan \Delta \lambda_{sv,n} \cot Az_{s,n}$$

$$\frac{\partial \Delta \lambda_{sv,n}}{\partial \phi'_{v,n}} = \tan \Delta \lambda_{sv,n} \tan \phi'_{v,n}$$

$$\frac{\partial R_{v,n}}{\partial \gamma'_n} = -R_{v,n} \tan \gamma'_n$$

$$\frac{\partial R_{v,n}}{\partial Az_{s,n}} = R_{v,n} \cot Az_{s,n}$$

$$\frac{\partial R_{v,n}}{\partial \phi'_{v,n}} = R_{v,n} \tan \phi'_{v,n}$$

$$\frac{\partial R_{v,n}}{\partial \Delta \lambda_{sv,n}} = -R_{v,n} \cot \Delta \lambda_{sv,n}$$

The partial derivatives given in equations (B17) and (B18) are

$$\frac{\partial Az_{g,n}}{\partial \phi_s} = \sin Az_{g,n} \tan \gamma_n$$

$$\frac{\partial Az_{g,n}}{\partial \lambda_s} = \sin \phi_s - \cos Az_{g,n} \cos \phi_s \tan \gamma_n$$

$$\frac{\partial \gamma_n}{\partial \phi_s} = \cos Az_{g,n}$$

$$\frac{\partial \gamma_n}{\partial \lambda_s} = \sin Az_{g,n} \cos \phi_s$$

If the partial derivatives of $\alpha_{n,i}$ with respect to $A_{n,i}$, $B_{n,i}$, and $C_{n,i}$ are obtained and substituted into expressions (B47), the results will become

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$$\left(\frac{\partial F}{\partial a}\right)_{n,i} = \frac{-\frac{\partial B_{n,i}}{\partial a} + \frac{B_{n,i}}{2A_{n,i}} \frac{\partial B_{n,i}}{\partial a} - 2\left(A_{n,i} \frac{\partial C_{n,i}}{\partial a} + C_{n,i} \frac{\partial A_{n,i}}{\partial a}\right) - 2 \tan \alpha_{n,i} \frac{\partial A_{n,i}}{\partial a}}{2A_{n,i} \tan \alpha_{n,i} + B_{n,i}} \cdot \frac{1}{2A_{n,i}(1 + \tan^2 \alpha_{n,i})}$$

$$\left(\frac{\partial F}{\partial f}\right)_{n,i} = \frac{-\frac{\partial B_{n,i}}{\partial f} + \frac{B_{n,i}}{2A_{n,i}} \frac{\partial B_{n,i}}{\partial f} - 2\left(A_{n,i} \frac{\partial C_{n,i}}{\partial f} + C_{n,i} \frac{\partial A_{n,i}}{\partial f}\right) - 2 \tan \alpha_{n,i} \frac{\partial A_{n,i}}{\partial f}}{2A_{n,i} \tan \alpha_{n,i} + B_{n,i}} \cdot \frac{1}{2A_{n,i}(1 + \tan^2 \alpha_{n,i})}$$

$$\left(\frac{\partial F}{\partial \phi'_{v,n}}\right)_{n,i} = \frac{-\frac{\partial B_{n,i}}{\partial \phi'_{v,n}} + \frac{B_{n,i}}{2A_{n,i}} \frac{\partial B_{n,i}}{\partial \phi'_{v,n}} - 2\left(A_{n,i} \frac{\partial C_{n,i}}{\partial \phi'_{v,n}} + C_{n,i} \frac{\partial A_{n,i}}{\partial \phi'_{v,n}}\right) - 2 \tan \alpha_{n,i} \frac{\partial A_{n,i}}{\partial \phi'_{v,n}}}{2A_{n,i} \tan \alpha_{n,i} + B_{n,i}} \cdot \frac{1}{2A_{n,i}(1 + \tan^2 \alpha_{n,i})}$$

$$\left(\frac{\partial F}{\partial R_{v,n}}\right)_{n,i} = \frac{-\frac{\partial B_{n,i}}{\partial R_{v,n}} + \frac{B_{n,i}}{2A_{n,i}} \frac{\partial B_{n,i}}{\partial R_{v,n}} - 2\left(A_{n,i} \frac{\partial C_{n,i}}{\partial R_{v,n}} + C_{n,i} \frac{\partial A_{n,i}}{\partial R_{v,n}}\right) - 2 \tan \alpha_{n,i} \frac{\partial A_{n,i}}{\partial R_{v,n}}}{2A_{n,i} \tan \alpha_{n,i} + B_{n,i}} \cdot \frac{1}{2A_{n,i}(1 + \tan^2 \alpha_{n,i})}$$

where

$$\begin{aligned} \frac{\partial A_{n,i}}{\partial a} = & -4a \left\{ \left(1 - \frac{a^2}{R_{v,n}^2}\right)(1-f)^4 + (2f-f^2)(1-f)^2 \sin^2 \phi'_{v,n} + \left[\frac{(2f-f^2)^2}{8} \sin^2 2\phi'_{v,n} \right. \right. \\ & \left. \left. + (2f-f^2)(1-f)^2 \left(1 - \frac{a^2}{R_{v,n}^2}\right) \cos^2 \phi'_{v,n}\right] \cos^2 \beta_i \right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial A_{n,i}}{\partial f} = & -4R_{v,n}^2(1-f)\left((1-f)^2\left(1-\frac{a^2}{R_{v,n}^2}\right)^2 - (2f-f^2)\sin^4\phi'_{v,n} - \left[1-2(2f-f^2)\right]\left(1-\frac{a^2}{R_{v,n}^2}\right)\sin^2\phi'_{v,n}\right. \\ & \left.- \left\{(2f-f^2)\left(1-\frac{a^2}{R_{v,n}^2}\right)\sin^2\phi'_{v,n} + \frac{1}{2}\left[1-2(2f-f^2)\right]\left(1-\frac{a^2}{R_{v,n}^2}\right)^2\right\}\cos^2\phi'_{v,n} \cos^2\beta_i\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial A_{n,i}}{\partial R_{v,n}} = & 2R_{v,n}\left\{\left(1-\frac{a^4}{R_{v,n}^4}\right)(1-f)^4 + (2f-f^2)\left[(2f-f^2)\sin^2\phi'_{v,n} + 2(1-f)^2\right]\sin^2\phi'_{v,n}\right. \\ & \left.+ (2f-f^2)\left[(2f-f^2)\sin^2\phi'_{v,n} + (1-f)^2\left(1-\frac{a^4}{R_{v,n}^4}\right)\right]\cos^2\phi'_{v,n} \cos^2\beta_i\right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial A_{n,i}}{\partial \phi'_{v,n}} = & 2R_{v,n}^2(2f-f^2)\left\{(2f-f^2)\sin^2\phi'_{v,n} + (1-f)^2\left(1-\frac{a^2}{R_{v,n}^2}\right) + \frac{1}{2}\left(1-\frac{a^2}{R_{v,n}^2}\right)\left[(2f-f^2)\cos^2\phi'_{v,n}\right.\right. \\ & \left.- (1-f)^2\left(1-\frac{a^2}{R_{v,n}^2}\right)\right]\cos^2\beta_i\left.\right\}\sin 2\phi'_{v,n} \end{aligned}$$

$$\frac{\partial B_{n,i}}{\partial a} = 2a(2f-f^2)\left[(2f-f^2)\sin^2\phi'_{v,n} + \left(1-2\frac{a^2}{R_{v,n}^2}\right)(1-f)^2\right]\sin 2\phi'_{v,n} \cos \beta_i$$

$$\frac{\partial B_{n,i}}{\partial f} = 2a^2(1-f)\left\{2(2f-f^2)\sin^2\phi'_{v,n} + \left[1-2(2f-f^2)\right]\left(1-\frac{a^2}{R_{v,n}^2}\right)\right\}\sin 2\phi'_{v,n} \cos \beta_i$$

$$\frac{\partial B_{n,i}}{\partial R_{v,n}} = 2(1-f)^2(2f-f^2)\frac{a^4}{R_{v,n}^3}\sin 2\phi'_{v,n} \cos \beta_i$$

$$\frac{\partial B_{n,i}}{\partial \phi'_{v,n}} = a^2\left\{(2f-f^2)\sin^2\phi'_{v,n} + 2\left[(2f-f^2)\sin^2\phi'_{v,n} + (1-f)^2\left(1-\frac{a^2}{R_{v,n}^2}\right)\right]\cos^2\phi'_{v,n}\right\}\cos \beta_i(2f-f^2)$$

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$$\frac{\partial c_{n,i}}{\partial a} = -2a \left\{ \left(1 - 2 \frac{a^2}{R_{v,n}^2} \right) (1 - f)^4 + \left[2 \left(1 - \frac{a^2}{R_{v,n}^2} \right) (1 - f)^2 + (2f - f^2) \sin^2 \phi'_{v,n} \right] (2f - f^2) \sin^2 \phi'_{v,n} \right\}$$

$$\frac{\partial c_{n,i}}{\partial f} = 2a^2 (1 - f) \left(2 \left(1 - \frac{a^2}{R_{v,n}^2} \right) (1 - f)^2 - \left[1 - 2(2f - f^2) \right] \left(2 - \frac{a^2}{R_{v,n}^2} \right) + 2(2f - f^2) \sin^2 \phi'_{v,n} \right) \sin^2 \phi'_{v,n}$$

$$\frac{\partial c_{n,i}}{\partial R_{v,n}} = -2 \frac{a^4}{R_{v,n}^3} (1 - f)^2 \left[(1 - f)^2 + (2f - f^2) \sin^2 \phi'_{v,n} \right]$$

$$\frac{\partial c_{n,i}}{\partial \phi'_{v,n}} = -a^2 (2f - f^2) \left[(1 - f)^2 \left(2 - \frac{a^2}{R_{v,n}^2} \right) + 2(2f - f^2) \sin^2 \phi'_{v,n} \right] \sin 2\phi'_{v,n}$$

Finally, it can be stated that since

$$e_{n,i} = F'(da, df, d\phi'_{v,n}, dR_{v,n})$$

and expressions for $d\phi$ and dR are given by equation (10) as functions of those relations given in (B37), (B41), and (B46), the coefficients of (13) can be readily obtained.

APPENDIX D

COEFFICIENTS OF THE DE GRAAFF-HUNTER EQUATIONS

The method of de Graaff-Hunter was chosen over that of Vening Meinesz since it is simpler to work with. This method is given in reference 1 on pages 127-130 in the form most widely used.

The accuracy of the two methods is largely dependent upon distance from the origin of the spheroid. For shorter distances the two methods will give accuracies of 0.001" of arc. For greater distances, the accuracy of the method of de Graaff-Hunter falls off whereas that of Vening Meinesz will give accuracies of 0.001" of arc or better.

The form of the de Graaff-Hunter equations as given in reference 1 is not satisfactory for this paper. The form of equation (9) is that form desired. To obtain it, the equations were expanded, and the corresponding coefficients of the five arbitrarily defined constants of a datum were collected. In so doing, it was determined that by making certain trigonometric substitutions the resulting coefficients can be simplified.

Further, in reference 1 δN_S and δN_O were considered positive when the new spheroid was above the old. In this paper h_S is positive for elevations above the spheroid. Since h_S and δN_S increase in opposite directions,

$$\Delta h_S = -\Delta \delta N_S$$

Thus, the expressions of the de Graaff-Hunter equations giving the change in spheroidal height must be multiplied by a minus one.

The coefficients of equation (9) are as follows:

$$L_a = \frac{(\cos u_S \sin u_O - \sin u_S \cos u_O \cos \omega)c}{a}$$

$$L_f = c \left[\sin 2u_S - \sin u_S \cos u_O \sin^2 u_O \cos \omega - \cos u_S \sin u_O (1 + \cos^2 u_O) \right]$$

$$L_\eta = c \sin u_S \sin \omega$$

$$L_\xi = -c (\sin u_S \sin u_O \cos \omega + \cos u_S \cos u_O)$$

$$L_{\delta N} = \frac{c (\sin u_S \cos u_O \cos \omega - \cos u_S \sin u_O)}{a}$$

APPENDIX D

$$M_a = - \left[\left(1 - \frac{h_s}{a} \right) \sec \phi_s \cos u_o \sin \omega \right] \frac{1}{a}$$

$$M_f = - \left(1 - \frac{h_s}{a} \right) \sec \phi_s \cos u_o \sin^2 u_o \sin \omega$$

$$M_\eta = - \left(1 - \frac{h_s}{a} \right) \sec \phi_s \cos \omega$$

$$M_\xi = - \left(1 - \frac{h_s}{a} \right) \sec \phi_s \sin u_o \sin \omega$$

$$M_{\delta N} = \left[\left(1 - \frac{h_s}{a} \right) \sec \phi_s \cos u_o \sin \omega \right] \frac{1}{a}$$

$$S_a = \cos u_s \cos \omega \cos u_o + \sin u_s \sin u_o - 1$$

$$S_f = -a \left[\sin u_s \sin u_o (1 + \cos^2 u_o) - \cos u_s \cos \omega \sin^2 u_o \cos u_o - \sin^2 u_s \right]$$

$$S_\eta = -a \cos u_s \sin \omega$$

$$S_\xi = -a (\sin u_s \cos u_s - \cos u_s \sin u_o \cos \omega)$$

$$S_{\delta N} = -(\cos u_s \cos \omega \cos u_o + \sin u_s \sin \omega_o)$$

where

$$c = \left(1 - \frac{h_s}{a} \right) (1 + f \cos^2 \phi_s)$$

$$\tan u_s = (1 - f) \tan \phi_s$$

$$\tan u_o = (1 - f) \tan \phi_s$$

and

$$\omega = \lambda_s - \lambda_o$$

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